CHAPTER 1

Angle Chasing

> This is your last chance. After this, there is no turning back. You take the blue pill—the story ends, you wake up in your bed and believe whatever you want to believe. You take the red pill—you stay in Wonderland and I show you how deep the rabbit-hole goes.
>
> Morpheus in *The Matrix*

Angle chasing is one of the most fundamental skills in olympiad geometry. For that reason, we dedicate the entire first chapter to fully developing the technique.

1.1 Triangles and Circles

Consider the following example problem, illustrated in Figure 1.1A.

Example 1.1. In quadrilateral $WXYZ$ with perpendicular diagonals (as in Figure 1.1A), we are given $\angle WZX = 30^\circ$, $\angle XWY = 40^\circ$, and $\angle WYZ = 50^\circ$.

(a) Compute $\angle Z$.
(b) Compute $\angle X$.

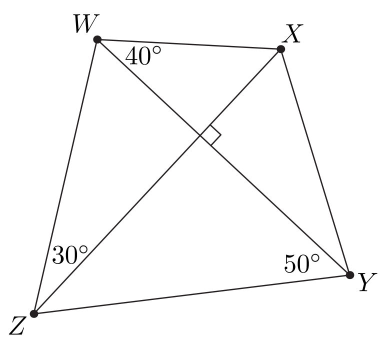

Figure 1.1A. Given these angles, which other angles can you compute?

You probably already know the following fact:

Proposition 1.2 (Triangle Sum). *The sum of the angles in a triangle is* 180°.

As it turns out, this is not sufficient to solve the entire problem, only the first half. The next section develops the tools necessary for the second half. Nevertheless, it is perhaps surprising what results we can derive from Proposition 1.2 alone. Here is one of the more surprising theorems.

Theorem 1.3 (Inscribed Angle Theorem). *If $\angle ACB$ is inscribed in a circle, then it subtends an arc with measure $2\angle ACB$.*

Proof. Draw in $\overline{OC}$. Set $\alpha = \angle ACO$ and $\beta = \angle BCO$, and let $\theta = \alpha + \beta$.

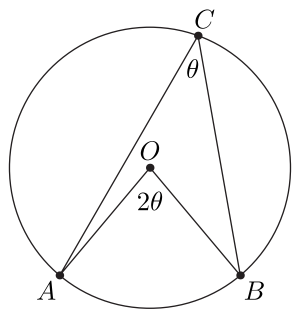

Figure 1.1B. The inscribed angle theorem.

We need some way to use the condition $AO = BO = CO$. How do we do so? Using isosceles triangles, roughly the only way we know how to convert lengths into angles. Because $AO = CO$, we know that $\angle OAC = \angle OCA = \alpha$. How does this help? Using Proposition 1.2 gives

$$\angle AOC = 180^\circ - (\angle OAC + \angle OCA) = 180^\circ - 2\alpha.$$

Now we do exactly the same thing with B. We can derive

$$\angle BOC = 180^\circ - 2\beta.$$

Therefore,

$$\angle AOB = 360^\circ - (\angle AOC + \angle BOC) = 360^\circ - (360^\circ - 2\alpha - 2\beta) = 2\theta$$

and we are done. □

We can also get information about the centers defined in Section 0.2. For example, recall the *incenter* is the intersection of the angle bisectors.

Example 1.4. If I is the incenter of $\triangle ABC$ then

$$\angle BIC = 90^\circ + \frac{1}{2}A.$$

Proof. We have

$$\begin{aligned}\angle BIC &= 180^\circ - (\angle IBC + \angle ICB) \\ &= 180^\circ - \frac{1}{2}(B+C) \\ &= 180^\circ - \frac{1}{2}(180^\circ - A) \\ &= 90^\circ + \frac{1}{2}A.\end{aligned}$$ $\square$

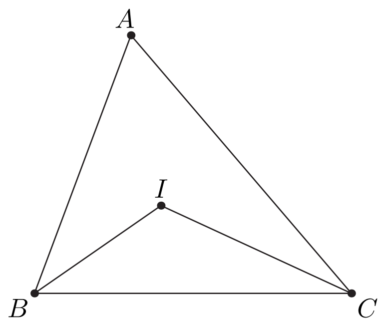

Figure 1.1C. The incenter of a triangle.

Problems for this Section

Problem 1.5. Solve the first part of Example 1.1. **Hint**: 185

Problem 1.6. Let ABC be a triangle inscribed in a circle ω. Show that $\overline{AC} \perp \overline{CB}$ if and only if $\overline{AB}$ is a diameter of ω.

Problem 1.7. Let O and H denote the circumcenter and orthocenter of an acute $\triangle ABC$, respectively, as in Figure 1.1D. Show that $\angle BAH = \angle CAO$. **Hints**: 540 373

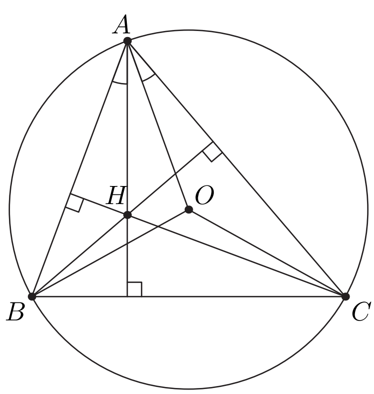

Figure 1.1D. The orthocenter and circumcenter. See Section 0.2 if you are not familiar with these.

1.2 Cyclic Quadrilaterals

The heart of this section is the following proposition, which follows directly from the inscribed angle theorem.

Proposition 1.8. *Let $ABCD$ be a convex cyclic quadrilateral. Then $\angle ABC + \angle CDA = 180^\circ$ and $\angle ABD = \angle ACD$.*

Here a **cyclic quadrilateral** refers to a quadrilateral that can be inscribed in a circle. See Figure 1.2A. More generally, points are **concyclic** if they all lie on some circle.

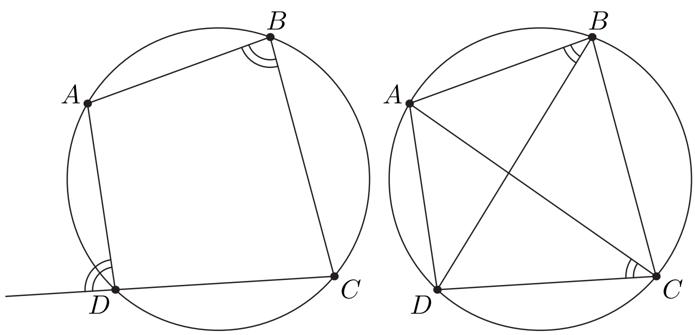

Figure 1.2A. Cyclic quadrilaterals with angles marked.

At first, this result seems not very impressive in comparison to our original theorem. However, it turns out that the converse of the above fact is true as well. Here it is more explicitly.

Theorem 1.9 (Cyclic Quadrilaterals). *Let $ABCD$ be a convex quadrilateral. Then the following are equivalent:*

(i) $ABCD$ *is cyclic.*
(ii) $\angle ABC + \angle CDA = 180^\circ$.
(iii) $\angle ABD = \angle ACD$.

This turns out to be extremely useful, and several applications appear in the subsequent sections. For now, however, let us resolve the problem we proposed at the beginning.

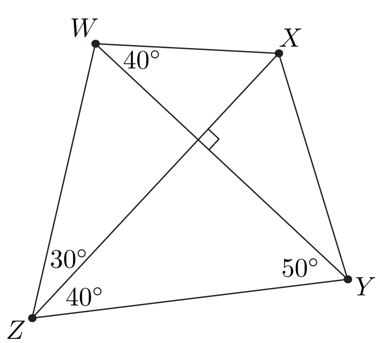

Figure 1.2B. Finishing Example 1.1. We discover $WXYZ$ is cyclic.

Solution to Example 1.1, part (b). Let P be the intersection of the diagonals. Then we have $\angle PZY = 90^\circ - \angle PYZ = 40^\circ$. Add this to the figure to obtain Figure 1.2B.

Now consider the 40° angles. They satisfy condition (iii) of Theorem 1.9. That means the quadrilateral $WXYZ$ is cyclic. Then by condition (ii), we know

$$\angle X = 180^\circ - \angle Z$$

Yet $\angle Z = 30^\circ + 40^\circ = 70^\circ$, so $\angle X = 110^\circ$, as desired. □

In some ways, this solution is totally unexpected. Nowhere in the problem did the problem mention a circle; nowhere in the solution does its center ever appear. And yet, using the notion of a cyclic quadrilateral reduced it to a mere calculation, whereas the problem was not tractable beforehand. This is where Theorem 1.9 draws its power.

We stress the importance of Theorem 1.9. It is not an exaggeration to say that more than 50% of standard olympiad geometry problems use it as an intermediate step. We will see countless applications of this theorem throughout the text.

Problems for this Section

Problem 1.10. Show that a trapezoid is cyclic if and only if it is isosceles.

Problem 1.11. Quadrilateral $ABCD$ has $\angle ABC = \angle ADC = 90^\circ$. Show that $ABCD$ is cyclic, and that $(ABCD)$ (that is, the circumcircle of $ABCD$) has diameter $\overline{AC}$.

1.3 The Orthic Triangle

In $\triangle ABC$, let D, E, F denote the feet of the altitudes from A, B, and C. The $\triangle DEF$ is called the **orthic triangle** of $\triangle ABC$. This is illustrated in Figure 1.3A.

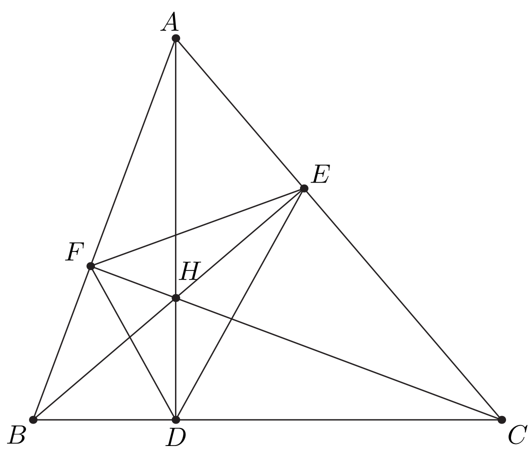

Figure 1.3A. The orthic triangle.

It also turns out that lines AD, BE, and CF all pass through a common point H, which is called the **orthocenter** of H. We will show the orthocenter exists in Chapter 3.

Although there are no circles drawn in the figure, the diagram actually contains six cyclic quadrilaterals.

Problem 1.12. In Figure 1.3A, there are six cyclic quadrilaterals with vertices in $\{A, B, C, D, E, F, H\}$. What are they? **Hint**: 91

To get you started, one of them is $AFHE$. This is because $\angle AFH = \angle AEH = 90^\circ$, and so we can apply (ii) of Theorem 1.9. Now find the other five!

Once the quadrilaterals are found, we are in a position of power; we can apply any part of Theorem 1.9 freely to these six quadrilaterals. (In fact, you can say even more—the right angles also tell you where the diameter of the circle is. See Problem 1.6.) Upon closer inspection, one stumbles upon the following.

Example 1.13. Prove that H is the incenter of $\triangle DEF$.

Check that this looks reasonable in Figure 1.3A.
We encourage the reader to try this problem before reading the solution below.

Solution to Example 1.13. Refer to Figure 1.3A. We prove that $\overline{DH}$ is the bisector of $\angle EDF$. The other cases are identical, and left as an exercise.

Because $\angle BFH = \angle BDH = 90^\circ$, we see that $BFHD$ is cyclic by Theorem 1.9. Applying the last clause of Theorem 1.9 again, we find

$$\angle FDH = \angle FBH.$$

Similarly, $\angle HEC = \angle HDC = 90^\circ$, so $CEHD$ is cyclic. Therefore,

$$\angle HDE = \angle HCE.$$

Because we want to prove that $\angle FDH = \angle HDE$, we only need to prove that $\angle FBH = \angle HCE$; in other words, $\angle FBE = \angle FCE$. This is equivalent to showing that $FBCE$ is cyclic, which follows from $\angle BFC = \angle BEC = 90^\circ$. (One can also simply show that both are equal to $90^\circ - A$ by considering right triangles BEA and CFA.)

Hence, $\overline{DH}$ is indeed the bisector, and therefore we conclude that H is the incenter of $\triangle DEF$. □

Combining the results of the above, we obtain our first configuration.

Lemma 1.14 (The Orthic Triangle). *Suppose $\triangle DEF$ is the orthic triangle of acute $\triangle ABC$ with orthocenter H. Then*

(a) Points A, E, F, H lie on a circle with diameter $\overline{AH}$.
(b) Points B, E, F, C lie on a circle with diameter $\overline{BC}$.
(c) H is the incenter of $\triangle DEF$.

Problems for this Section

Problem 1.15. Work out the similar cases in the solution to Example 1.13. That is, explicitly check that $\overline{EH}$ and $\overline{FH}$ are actually bisectors as well.

Problem 1.16. In Figure 1.3A, show that $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ are each similar to $\triangle ABC$. **Hint**: 181

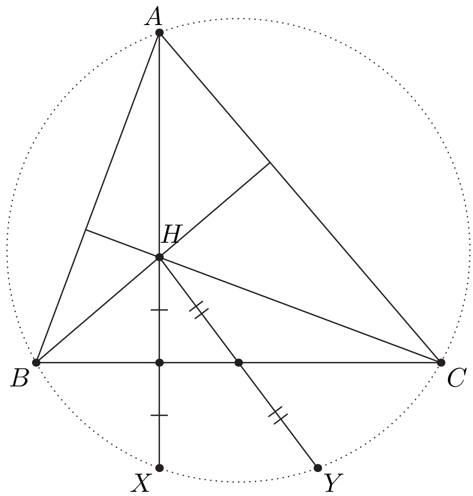

Figure 1.3B. Reflecting the orthocenter. See Lemma 1.17.

Lemma 1.17 (Reflecting the Orthocenter). *Let H be the orthocenter of $\triangle ABC$, as in Figure 1.3B. Let X be the reflection of H over $\overline{BC}$ and Y the reflection over the midpoint of $\overline{BC}$.*

(a) Show that X lies on (ABC).
(b) Show that $\overline{AY}$ is a diameter of (ABC). **Hint**: 674

1.4 The Incenter/Excenter Lemma

We now turn our attention from the orthocenter to the incenter. Unlike before, the cyclic quadrilateral is essentially given to us. We can use it to produce some interesting results.

Lemma 1.18 (The Incenter/Excenter Lemma). *Let ABC be a triangle with incenter I. Ray AI meets (ABC) again at L. Let I_A be the reflection of I over L. Then,*

(a) The points I, B, C, and I_A lie on a circle with diameter $\overline{II_A}$ and center L. In particular, $LI = LB = LC = LI_A$.
(b) Rays BI_A and CI_A bisect the exterior angles of $\triangle ABC$.

By "exterior angle", we mean that ray BI_A bisects the angle formed by the segment BC and the extension of line AB past B. The point I_A is called the **A-excenter*** of $\triangle ABC$; we visit it again in Section 2.6.

Let us see what we can do with cyclic quadrilateral $ABLC$.

* Usually the A-excenter is defined as the intersection of exterior angle bisectors of $\angle B$ and $\angle C$, rather than as the reflection of I over L. In any case, Lemma 1.18 shows these definitions are equivalent.

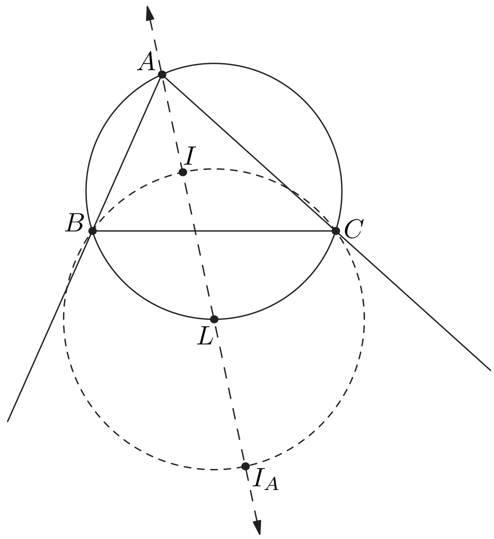

Figure 1.4A. Lemma 1.18, the incenter/excenter lemma.

Proof. Let $\angle A = 2\alpha$, $\angle B = 2\beta$, and $\angle C = 2\gamma$ and notice that $\angle A + \angle B + \angle C = 180^\circ \Rightarrow \alpha + \beta + \gamma = 90^\circ$.

Our first goal is to prove that $LI = LB$. We prove this by establishing $\angle IBL = \angle LIB$ (this lets us convert the conclusion completely into the language of angles). To do this, we invoke (iii) of Theorem 1.9 to get $\angle CBL = \angle LAC = \angle IAC = \alpha$. Therefore,

$$\angle IBL = \angle IBC + \angle CBL = \beta + \alpha.$$

All that remains is to compute $\angle BIL$. But this is simple, as

$$\angle BIL = 180^\circ - \angle AIB = \angle IBA + \angle BAI = \alpha + \beta$$

Therefore triangle LBI is isosceles, with $LI = LB$, which is what we wanted.

Similar calculations give $LI = LC$.

Because $LB = LI = LC$, we see that L is indeed the center of (IBC). Because L is given to be the midpoint of $\overline{II_A}$, it follows that $\overline{II_A}$ is a diameter of (LBC) as well.

Let us now approach the second part. We wish to show that $\angle I_A BC = \frac{1}{2}(180^\circ - 2\beta) = 90^\circ - \beta$. Recalling that $\overline{II_A}$ is a diameter of the circle, we observe that

$$\angle IBI_A = \angle ICI_A = 90^\circ.$$

so $\angle I_A BC = \angle I_A BI - \angle IBC = 90^\circ - \beta$.

Similar calculations yield that $\angle BCI_A = 90^\circ - \gamma$, as required. □

This configuration shows up very often in olympiad geometry, so recognize it when it appears!

Problem for this Section

Problem 1.19. Fill in the two similar calculations in the proof of Lemma 1.18.

1.5 Directed Angles

Some motivation is in order. Look again at Figure 1.3A. We assumed that $\triangle ABC$ was acute. What happens if that is not true? For example, what if $\angle A > 90^\circ$ as in Figure 1.5A?

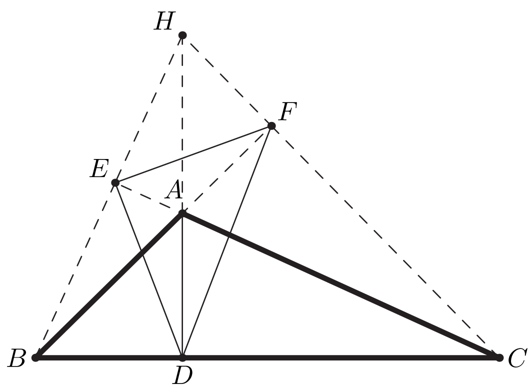

Figure 1.5A. No one likes configuration issues.

There should be something scary in the above figure. Earlier, we proved that points B, E, A, D were concyclic using criterion (iii) of Theorem 1.9. Now, the situation is different. Has anything changed?

Problem 1.20. Recall the six cyclic quadrilaterals from Problem 1.12. Check that they are still cyclic in Figure 1.5A.

Problem 1.21. Prove that, in fact, A is the orthocenter of $\triangle HBC$.

In this case, we are okay, but the dangers are clear. For example, when $\triangle ABC$ was acute, we proved that B, H, F, D were concyclic by noticing that the opposite angles satisfied $\angle BDH + \angle HFB = 180^\circ$. Here, however, we instead have to use the fact that $\angle BDH = \angle BFH$; in other words, for the same problem we have to use different parts of Theorem 1.9. We should not need to worry about solving the same problem twice!

How do we handle this? The solution is to use **directed angles** mod 180°. Such angles will be denoted with a $\measuredangle$ symbol instead of the standard $\angle$. (This notation is not standard; should you use it on a contest, do not neglect to say so in the opening lines of your solution.)

Here is how it works. First, we consider $\measuredangle ABC$ to be *positive* if the vertices A, B, C appear in clockwise order, and *negative* otherwise. In particular, $\measuredangle ABC \neq \measuredangle CBA$; they are negatives. See Figure 1.5B.

Then, we are taking the angles modulo 180°. For example,

$$-150^\circ = 30^\circ = 210^\circ.$$

Why on earth would we adopt such a strange convention? The key is that our Theorem 1.9 can now be rewritten as follows.

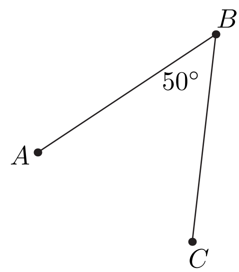

Figure 1.5B. Here, $\measuredangle ABC = 50^\circ$ and $\measuredangle CBA = -50^\circ$.

Theorem 1.22 (Cyclic Quadrilaterals with Directed Angles). *Points A, B, X, Y lie on a circle if and only if*

$$\measuredangle AXB = \measuredangle AYB.$$

This seems too good to be true, as we have dropped the convex condition—there is now only one case of the theorem. In other words, as long as we direct our angles, we no longer have to worry about configuration issues when applying Theorem 1.9.

Problem 1.23. Verify that parts (ii) and (iii) of Theorem 1.9 match the description in Theorem 1.22.

We present some more convenient truths in the following proposition.

Proposition 1.24 (Directed Angles). *For any distinct points A, B, C, P in the plane, we have the following rules.*

Oblivion. $\measuredangle APA = 0$.

Anti-Reflexivity. $\measuredangle ABC = -\measuredangle CBA$.

Replacement. *$\measuredangle PBA = \measuredangle PBC$ if and only if A, B, C are collinear. (What happens when $P = A$?) Equivalently, if C lies on line BA, then the A in $\measuredangle PBA$ may be replaced by C.*

Right Angles. *If $\overline{AP} \perp \overline{BP}$, then $\measuredangle APB = \measuredangle BPA = 90^\circ$.*

Directed Angle Addition. $\measuredangle APB + \measuredangle BPC = \measuredangle APC$.

Triangle Sum. $\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0$.

Isosceles Triangles. *$AB = AC$ if and only if $\measuredangle ACB = \measuredangle CBA$.*

Inscribed Angle Theorem. *If (ABC) has center P, then $\measuredangle APB = 2\measuredangle ACB$.*

Parallel Lines. *If $\overline{AB} \parallel \overline{CD}$, then $\measuredangle ABC + \measuredangle BCD = 0$.*

One thing we have to be careful about is that $2\measuredangle ABC = 2\measuredangle XYZ$ does not imply $\measuredangle ABC = \measuredangle XYZ$, because we are taking angles modulo 180°. Hence it does not make sense to take half of a directed angle.[†]

Problem 1.25. Convince yourself that all the claims in Proposition 1.24 are correct.

[†] Because of this, it is customary to take arc measures modulo 360°. We may then write the inscribed angle theorem as $\measuredangle ABC = \frac{1}{2}\widehat{AC}$. This is okay since $\measuredangle ABC$ is taken mod 180° but $\widehat{AC}$ is taken mod 360°.

Directed angles are quite counterintuitive at first, but with a little practice they become much more natural. The right way to think about them is to solve the problem for a specific configuration, but write down all statements in terms of directed angles. The solution for a specific configuration then automatically applies to all configurations.

Before moving in to a less trivial example, let us finish the issue with the orthic triangle once and for all.

Example 1.26. Let H be the orthocenter of $\triangle ABC$, acute or not. Using directed angles, show that $AEHF$, $BFHD$, $CDHE$, $BEFC$, $CFDA$, and $ADEB$ are cyclic.

Solution. We know that

$$90^\circ = \measuredangle ADB = \measuredangle ADC$$
$$90^\circ = \measuredangle BEC = \measuredangle BEA$$
$$90^\circ = \measuredangle CFA = \measuredangle CFB$$

because of right angles. Then

$$\measuredangle AEH = \measuredangle AEB = -\measuredangle BEA = -90^\circ = 90^\circ$$

and

$$\measuredangle AFH = \measuredangle AFC = -\measuredangle CFA = -90^\circ = 90^\circ$$

so A, E, F, H are concyclic. Also,

$$\measuredangle BFC = -\measuredangle CFB = -90^\circ = 90^\circ = \measuredangle BEC$$

so B, E, F, C are concyclic. The other quadrilaterals have similar stories. □

We conclude with one final example.

Lemma 1.27 (Miquel Point of a Triangle). *Points D, E, F lie on lines BC, CA, and AB of $\triangle ABC$, respectively. Then there exists a point lying on all three circles (AEF), (BFD), (CDE).*

This point is often called the **Miquel point** of the triangle.

It should be clear by looking at Figure 1.5C that many, many configurations are possible. Trying to handle this with standard angles would be quite messy. Fortunately, we can get them all in one go with directed angles.

Let K be the intersection of (BFD) and (CDE) other than D. The goal is to show that $AFEK$ is cyclic as well. For the case when K is inside $\triangle ABC$, this is an easy angle chase. All we need to do is use the corresponding statements with directed angles for each step.

We strongly encourage readers to try this themselves before reading the solution that follows.

First, here is the solution for the first configuration of Figure 1.5C. Define K as above. Now we just notice that $\angle FKD = 180^\circ - B$ and $\angle EKD = 180^\circ - C$. Consequently, $\angle FKE = 360^\circ - (180^\circ - C) - (180^\circ - B) = B + C = 180^\circ - A$ and $AFEK$ is cyclic. Now we just need to convert this into directed angles.

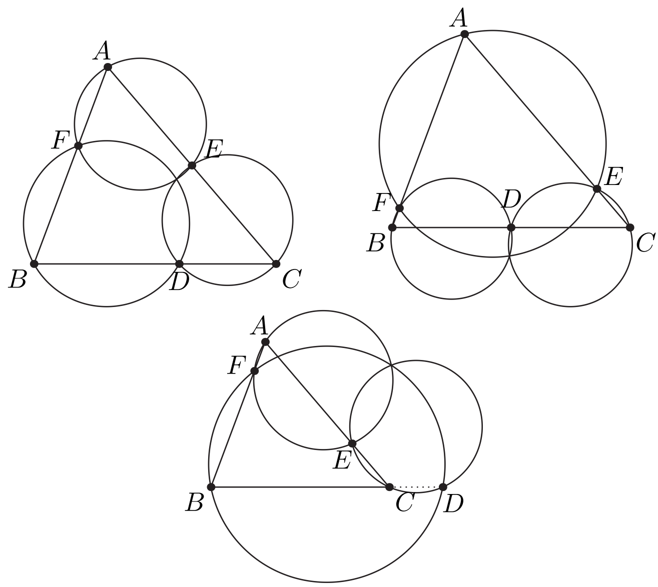

Figure 1.5C. The Miquel point, as in Lemma 1.27.

Proof. The first two claims are just

$$\measuredangle FKD = \measuredangle FBD = \measuredangle ABC \text{ and } \measuredangle DKE = \measuredangle DCE = \measuredangle BCA.$$

We also know that

$$\measuredangle FKD + \measuredangle DKE + \measuredangle EKF = 0 \text{ and } \measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0.$$

The first equation represents the fact that the sum of the angles at K is 360°; the second is the fact that the sum of the angles in a triangle is 180°. From here we derive that $\measuredangle CAB = \measuredangle EKF$. But $\measuredangle CAB = \measuredangle EAF$; hence $\measuredangle EAF = \measuredangle EKF$ as desired. □

Having hopefully convinced you that directed angles are natural and often useful, let us provide a warning on when not to use them. Most importantly, you should not use directed angles when the problem only works for a certain configuration! An example of this is Problem 1.38; the problem statement becomes false if the quadrilateral is instead $ABDC$. You should also avoid using directed angles if you need to invoke trigonometry, or if you need to take half an angle (as in Problem 1.38 again). These operations do not make sense modulo 180°.

Problems for this Section

Problem 1.28. We claimed that $\measuredangle FKD + \measuredangle DKE + \measuredangle EKF = 0$ in the above proof. Verify this using Proposition 1.24.

Problem 1.29. Show that for any distinct points A, B, C, D we have $\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB = 0$. **Hints**: 114 645

Lemma 1.30. *Points A, B, C lie on a circle with center O. Show that $\measuredangle OAC = 90^\circ - \measuredangle CBA$. (This is not completely trivial.)* **Hints**: 8 530 109

1.6 Tangents to Circles and Phantom Points

Here we introduce one final configuration and one general technique.

First, we discuss the **tangents** to a circle. In many ways, one can think of it as Theorem 1.22 applied to the "quadrilateral" $AABC$. Indeed, consider a point X on the circle and the line XA. As we move X closer to A, the line XA approaches the tangent at A. The limiting case becomes the theorem below.

Proposition 1.31 (Tangent Criterion). *Suppose $\triangle ABC$ is inscribed in a circle with center O. Let P be a point in the plane. Then the following are equivalent:*

(i) $\overline{PA}$ *is tangent to* (ABC).
(ii) $\overline{OA} \perp \overline{AP}$.
(iii) $\measuredangle PAB = \measuredangle ACB$.

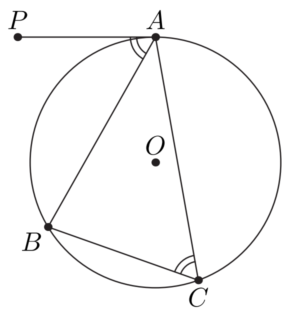

Figure 1.6A. PA is a tangent to (ABC). See Proposition 1.31.

In the following example we also introduce the technique of adding a **phantom point**. (This general theme is sometimes also called **reverse reconstruction**.)

Example 1.32. Let ABC be an acute triangle with circumcenter O, and let K be a point such that $\overline{KA}$ is tangent to (ABC) and $\angle KCB = 90^\circ$. Point D lies on $\overline{BC}$ such that $\overline{KD} \parallel \overline{AB}$. Show that line $\overline{DO}$ passes through A.

This problem is perhaps a bit trickier to solve directly, because we have not developed any tools to show that three points are collinear. (We will!) But here is a different idea. We define a phantom point D' as the intersection of ray AO with $\overline{BC}$. If we can show that $\overline{KD'} \parallel \overline{AB}$, then this will prove $D' = D$, because there is only one point on $\overline{BC}$ with $\overline{KD} \parallel \overline{AB}$.

Fortunately, this can be done with merely the angle chasing that we know earlier. We leave it as Problem 1.33. As a hint, you will have to use both parts of Proposition 1.31.

We have actually encountered a similar idea before, in our proof of Lemma 1.27. The idea was to let (BDF) and (CDE) intersect at a point K, and then show that K was on the

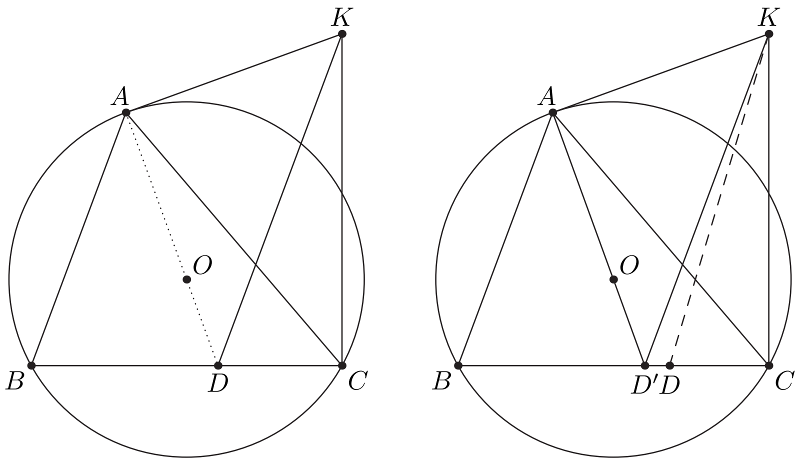

Figure 1.6B. Example 1.32, and the phantom point.

third circle as well. This theme is common in geometry. A second example where phantom points are helpful is Lemma 1.45 on page 19.

It is worth noting that solutions using phantom points can often (but not always) be rearranged to avoid them, although such solutions may be much less natural. For example, another way to solve Example 1.32 is to show that $\measuredangle KAO = \measuredangle KAD$. Problem 1.34 is the most common example of a problem that is not easy to rewrite without phantom points.

Problems for this Section

Problem 1.33. Let ABC be a triangle and let ray AO meet $\overline{BC}$ at D'. Point K is selected so that $\overline{KA}$ is tangent to (ABC) and $\angle KC = 90^\circ$. Prove that $\overline{KD'} \parallel \overline{AB}$.

Problem 1.34. In scalene triangle ABC, let K be the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of $\overline{BC}$. Prove that the points A, B, C, K are concyclic.
Hints: 356 101

1.7 Solving a Problem from the IMO Shortlist

To conclude the chapter, we leave the reader with one last example problem. We hope the discussion is instructive.

Example 1.35 (Shortlist 2010/G1). Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

In this problem there are two possible configurations. Directed angles allows us to handle both, but let us focus on just one—say P_2 and Q_2.

The first thing we notice is the orthic triangle. Because of it we should keep the results of Lemma 1.14 close at heart. Additionally, we are essentially given that $ACBP_2$ is a cyclic

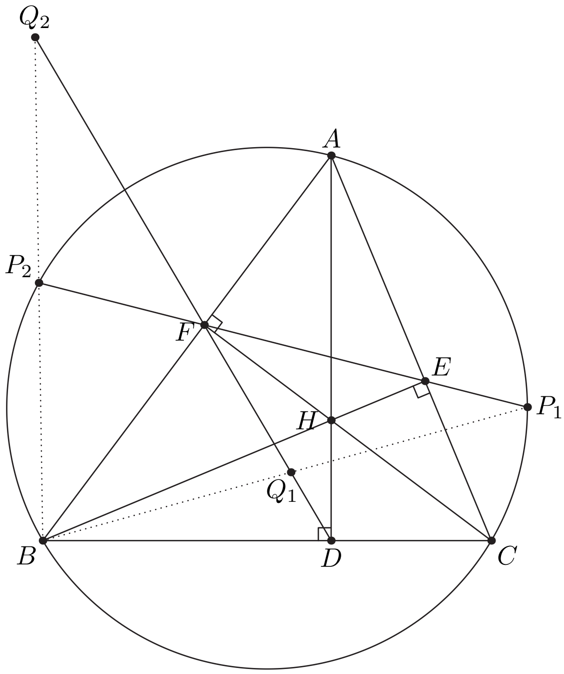

Figure 1.7A. IMO Shortlist 2010, Problem G1 (Example 1.35).

quadrilateral. Let us see what we can do with that. The conclusion $AP_2 = AQ_2$ seems better expressed in terms of angles—we want to show that $\measuredangle AQ_2P_2 = \measuredangle Q_2P_2A$. Now we already know $\measuredangle Q_2P_2A$, because

$$\measuredangle Q_2P_2A = \measuredangle BP_2A = \measuredangle BCA$$

so it is equivalent to compute $\measuredangle AQ_2P_2$.

There are two ways to realize the next step. The first is wishful thinking—the hope that a convenient cyclic quadrilateral will give us $\measuredangle AQ_2P_2$. The second way is to have a scaled diagram at hand. Either way, we stumble upon the following hope: might AQ_2P_2F be cyclic? It certainly looks like it in the diagram.

How might we prove that AQ_2P_2F is cyclic? Trying to use supplementary angles seems not as hopeful, because this is what we want to use as a final step. However, inscribed arcs seems more promising. We already know $\measuredangle AP_2Q_2 = \measuredangle ACB$. Might we be able to find AFQ_2? Yes—we know that

$$\measuredangle AFQ_2 = \measuredangle AFD$$

and now we are certain this will succeed, because $\measuredangle AFD$ is entirely within the realm of $\triangle ABC$ and its orthic triangle. In other words, we have eliminated P and Q. In fact,

$$\measuredangle AFD = \measuredangle ACD = \measuredangle ACB$$

since $AFDC$ is cyclic. This solves the problem for P_2 and Q_2. Because we have been careful to direct all the angles, this automatically solves the case P_1 and Q_1 as well—and this is why directed angles are useful.

It is important to realize that the above is not a well-written proof, but instead a description of how to arrive at the solution. Below is an example of how to write the proof in a contest—one direction only (so without working backwards like we did at first), and without the motivation. Follow along in the following proof with P_1 and Q_1, checking that the directed angles work out.

Solution to Example 1.35. First, because $APBC$ and $AFDC$ are cyclic,

$$\measuredangle QPA = \measuredangle BPA = \measuredangle BCA = \measuredangle DCA = \measuredangle DFA = \measuredangle QFA.$$

Therefore, we see $AFPQ$ is cyclic. Then

$$\measuredangle AQP = \measuredangle AFP = \measuredangle AFE = \measuredangle AHE = \measuredangle DHE = \measuredangle DCE = \measuredangle BCA.$$

We deduce that $\measuredangle AQP = \measuredangle BCA = \measuredangle QPA$ which is enough to imply that $\triangle APQ$ is isosceles with $AP = AQ$. $\square$

This problem is much easier if Lemma 1.14 is kept in mind. In that case, the only key observation is that $AFPQ$ is cyclic. As we saw above, one way to make this key observation is to merely peruse the diagram for quadrilaterals that appear cyclic. That is why it is often a good idea, on any contest problem, to draw a scaled diagram using ruler and compass—in fact, preferably more than one diagram. This often gives away intermediate steps in the problem, prevents you from missing obvious facts, or gives you something to attempt to prove. It will also prevent you from wasting time trying to prove false statements.

1.8 Problems

Problem 1.36. Let $ABCDE$ be a convex pentagon such that $BCDE$ is a square with center O and $\angle A = 90^\circ$. Prove that $\overline{AO}$ bisects $\angle BAE$. **Hints**: 18 115 **Sol:** p.241

Problem 1.37 (BAMO 1999/2). Let $O = (0, 0)$, $A = (0, a)$, and $B = (0, b)$, where $0 < a < b$ are reals. Let Γ be a circle with diameter $\overline{AB}$ and let P be any other point on Γ. Line PA meets the x-axis again at Q. Prove that $\angle BQP = \angle BOP$. **Hints**: 635 100

Problem 1.38. In cyclic quadrilateral $ABCD$, let I_1 and I_2 denote the incenters of $\triangle ABC$ and $\triangle DBC$, respectively. Prove that I_1I_2BC is cyclic. **Hints**: 684 569

Problem 1.39 (CGMO 2012/5). Let ABC be a triangle. The incircle of $\triangle ABC$ is tangent to $\overline{AB}$ and $\overline{AC}$ at D and E respectively. Let O denote the circumcenter of $\triangle BCI$.

Prove that $\angle ODB = \angle OEC$. **Hints**: 643 89 **Sol:** p.241

Problem 1.40 (Canada 1991/3). Let P be a point inside circle ω. Consider the set of chords of ω that contain P. Prove that their midpoints all lie on a circle. **Hints**: 455 186 169

Problem 1.41 (Russian Olympiad 1996). Points E and F are on side $\overline{BC}$ of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$. **Hints**: 245 614

Lemma 1.42. *Let ABC be an acute triangle inscribed in circle Ω. Let X be the midpoint of the arc $\widehat{BC}$ not containing A and define Y, Z similarly. Show that the orthocenter of XYZ is the incenter I of ABC.* **Hints**: 432 21 326 195

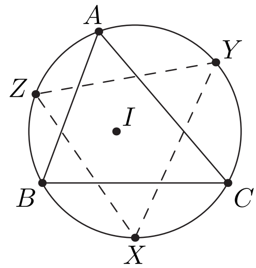

Figure 1.8A. Lemma 1.42. I is the orthocenter of $\triangle XYZ$.

Problem 1.43 (JMO 2011/5). Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$.

Prove that $\overline{BE}$ bisects $\overline{AC}$. **Hints**: 401 575 **Sol:** p.242

Lemma 1.44 (Three Tangents). *Let ABC be an acute triangle. Let $\overline{BE}$ and $\overline{CF}$ be altitudes of $\triangle ABC$, and denote by M the midpoint of $\overline{BC}$. Prove that $\overline{ME}$, $\overline{MF}$, and the line through A parallel to $\overline{BC}$ are all tangents to (AEF).* **Hints**: 24 335

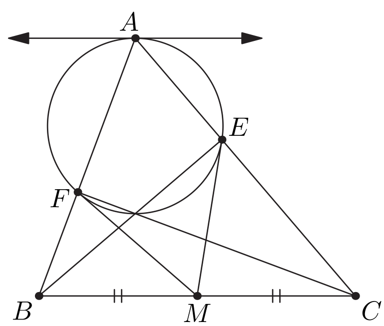

Figure 1.8B. Lemma 1.44, involving tangents to (AEF).

Lemma 1.45 (Right Angles on Incircle Chord). *The incircle of $\triangle ABC$ is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let M and N be the midpoints of $\overline{BC}$ and $\overline{AC}$, respectively. Ray BI meets line EF at K. Show that $\overline{BK} \perp \overline{CK}$. Then show K lies on line MN.* **Hints**: 460 84

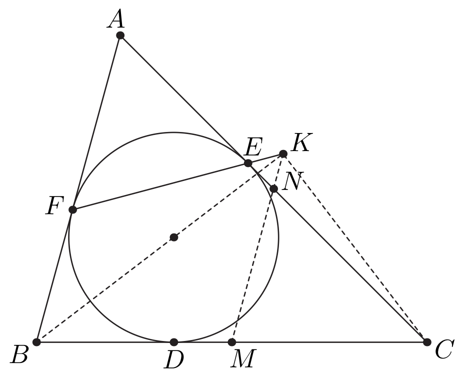

Figure 1.8C. Diagram for Lemma 1.45.

Problem 1.46 (Canada 1997/4). The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180^\circ$. Prove that $\angle OBC = \angle ODC$. **Hints**: 386 110 214 **Sol:** p.242

Problem 1.47 (IMO 2006/1). Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$ and that equality holds if and only if $P = I$. **Hints**: 212 453 670

Lemma 1.48 (Simson Line). *Let ABC be a triangle and P be any point on (ABC). Let X, Y, Z be the feet of the perpendiculars from P onto lines BC, CA, and AB. Prove that points X, Y, Z are collinear.* **Hints**: 278 502 **Sol:** p.243

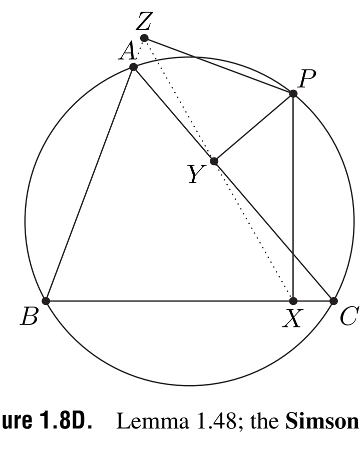

Figure 1.8D. Lemma 1.48; the **Simson line**.

Problem 1.49 (USAMO 2010/1). Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB. **Hint**: 661

Problem 1.50 (IMO 2013/4). Let ABC be an acute triangle with orthocenter H, and let W be a point on the side $\overline{BC}$, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. ω_1 is the circumcircle of triangle BWN and X is a point such that $\overline{WX}$ is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that $\overline{WY}$ is a diameter of ω_2. Show that the points X, Y, and H are collinear. **Hints**: 106 157 15 **Sol:** p.243

Problem 1.51 (IMO 1985/1). A circle has center on the side $\overline{AB}$ of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$. **Hints**: 36 201