CHAPTER 7

Barycentric Coordinates

> I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.
> Maslow's Hammer

We now present another technique, barycentric coordinates. At the time of writing, it is surprisingly unknown to most olympiad contestants and problem writers.

In this chapter, the area notation $[XYZ]$ refers to signed areas (see Section 5.1). That means that the area $[XYZ]$ is positive if the points X, Y, Z are oriented in counterclockwise order, and negative otherwise.

7.1 Definitions and First Theorems

Throughout this section we fix a nondegenerate triangle ABC, called the **reference triangle**. (This is much like selecting an origin and axes in a Cartesian coordinate system.) Each point P in the plane is assigned an ordered triple of real numbers $P = (x, y, z)$ such that

$$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C} \quad \text{and} \quad x + y + z = 1.$$

These are called the **barycentric coordinates** of point P with respect to triangle ABC.

Barycentric coordinates are also sometimes called **areal coordinates** because if $P = (x, y, z)$, then the signed area $[PBC]$ is equal to $x[ABC]$, and so on. In other words, these coordinates can be viewed as

$$P = \left(\frac{[PBC]}{[ABC]}, \frac{[PCA]}{[BCA]}, \frac{[PAB]}{[CAB]} \right).$$

The areas are signed in order to permit the point P to lie outside the triangle. If $P = (x, y, z)$ and A lie on opposite sides of $\overline{BC}$, then the signed areas of $[PBC]$ and $[ABC]$ have opposite signs and $x < 0$. In particular, the point P lies in the interior of ABC if and only if $x, y, z > 0$.

Observe that $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$. This is why barycentric coordinates are substantially more suited for standard triangle geometry problems; the vertices are both simple and symmetric.

The soul of barycentric coordinates derives from the following result, which we state without proof.

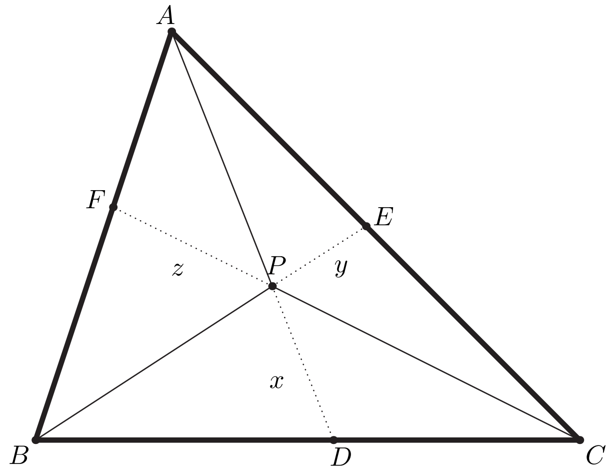

Figure 7.1A. Regions corresponding to the areas of ABC when P is inside the triangle.

Theorem 7.1 (Barycentric Area Formula). *Let P_1, P_2, P_3 be points with barycentric coordinates $P_i = (x_i, y_i, z_i)$ for $i = 1, 2, 3$. Then the signed area of $\triangle P_1P_2P_3$ is given by the determinant*

$$\frac{[P_1P_2P_3]}{[ABC]} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Again, the area is signed, following the convention in Section 5.1.

As a corollary, we derive the equation of a line.

Theorem 7.2 (Equation of a Line). *The equation of a line takes the form $ux + vy + wz = 0$ where u, v, w are real numbers. The u, v, and w are unique up to scaling.*

Proof. The main idea is that three points are collinear if and only if the signed area of their "triangle" is zero. Suppose we wish to characterize the points $P = (x, y, z)$ lying on a line XY, where $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2)$. Using the above area formula with $[PAB] = 0$, we find this occurs precisely when

$$0 = (y_1z_2 - y_2z_1)x + (z_1x_2 - z_2x_1)y + (x_1y_2 - x_2y_1)z,$$

i.e., $0 = ux + vy + wz$ for some constants u, v, w. $\square$

In particular, the equation for the line AB is simply $z = 0$, by substituting $(1, 0, 0)$ and $(0, 1, 0)$ into $ux + vy + wz = 0$. In general, the formula for a cevian through A is of the form $vy + wz = 0$, by substituting the point $A = (1, 0, 0)$.

In fact, the above techniques are already sufficient to prove both Ceva's and Menelaus's theorem.

Example 7.3 (Ceva's Theorem). Let D, E, F be points in the interiors of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of a triangle ABC. Then the cevians $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are concurrent if and only if

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = 1.$$

Proof. Define

$$D = (0, d, 1-d)$$
$$E = (1-e, 0, e)$$
$$F = (f, 1-f, 0)$$

where d, e, f are real numbers strictly between 0 and 1.

Then the corresponding equations of lines are

$$\overline{AD} : dz = (1-d)y$$
$$\overline{BE} : ex = (1-e)z$$
$$\overline{CF} : fy = (1-f)x.$$

We wish to show there is a nontrivial solution to this system of equations (i.e., one other than (0, 0, 0)) if and only if $def = (1-d)(1-e)(1-f)$, which is evidently equivalent to the constraint $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = 1$.

First suppose that a nontrivial solution (x, y, z) exists. Notice that if any of x, y, z is zero, then the others must all be zero as well. So we may assume $xyz \neq 0$. Now taking the product and cancelling xyz yields $def = (1-d)(1-e)(1-f)$.

On the other hand, suppose the condition $def = (1-d)(1-e)(1-f)$ holds. We opportunistically pick x, y, z. Put $y_1 = d$ and $z_1 = 1-d$. Then we require

$$x_1 = \frac{1-e}{e}(1-d) = \frac{f}{1-f}d$$

and this is okay since $def = (1-d)(1-e)(1-f)$; hence we can set x_1 as above. Thus $x = x_1$, $y = y_1$, and $z = z_1$ is a solution to the equations above.

However, there is no reason to believe that $x_1 + y_1 + z_1 = 1$, so the triple we found earlier may not actually correspond to a point. (However, we at least know $x_1, y_1, z_1 > 0$.) This is not a big issue: we instead consider the triple

$$(x, y, z) = \left(\frac{x_1}{x_1+y_1+z_1}, \frac{y_1}{x_1+y_1+z_1}, \frac{z_1}{x_1+y_1+z_1}\right)$$

which still satisfies the conditions, but now has sum 1. Thus this triple corresponds to the desired point of concurrency. $\square$

The last step in the above proof illustrates that barycentric coordinates are *homogeneous*. Let us make his idea explicit. Suppose (x, y, z) lies on a line

$$ux + vy + wz = 0.$$

Then so does the "triple", $(2x, 2y, 2z)$, $(1000x, 1000y, 1000z)$ or indeed any multiple. In light of this, we permit **unhomogenized barycentric coordinates** by writing $(x : y : z)$ as shorthand for the appropriate triple

$$(x : y : z) = \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right)$$

whenever $x + y + z \neq 0$. Note the use of colons instead of commas. An equivalent definition is as follows: for any nonzero k, the points $(x : y : z)$ and $(kx : ky : kz)$ are considered the same, and $(x : y : z) = (x, y, z)$ when $x + y + z = 1$.

This shorthand is convenient because such coordinates may still be "plugged in" to the line formula, often saving computations. For example, we have the following convenient corollary.

Theorem 7.4 (Barycentric Cevian). *Let $P = (x_1 : y_1 : z_1)$ be any point other than A. Then the points on line AP (other than A) can be parametrized by*

$$(t : y_1 : z_1)$$

where $t \in \mathbb{R}$ and $t + y_1 + z_1 \neq 0$.

On the other hand, it makes no sense to put unhomogenized coordinates into, say, the area formula. For these purposes, our usual coordinates (x, y, z) with the restriction $x + y + z = 1$ will be called **homogenized barycentric coordinates** and delimited with colons.

Problems for this Section

Problem 7.5. Find the coordinates for the midpoint of $\overline{AB}$. **Hint**: 623

Lemma 7.6 (Barycentric Conjugates). *Let $P = (x : y : z)$ be a point with $x, y, z \neq 0$. Show that the isogonal conjugate of P is given by*

$$P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right)$$

and the isotomic conjugate is given by

$$P^t = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right).$$

Hint: 419

7.2 Centers of the Triangle

In Table 7.1 we give explicit forms for several centers of the reference triangle. Remember that $(u : v : w)$ refers to the point with coordinates $(\frac{u}{u+v+w}, \frac{v}{u+v+w}, \frac{w}{u+v+w})$; that is, we are not normalizing the coordinates.

This is so important we say it twice: *the coordinates here are unhomogenized.*

Here G, I, H, O denote the usual centroid, incenter, orthocenter, and circumcenter, while I_A denotes the A-excenter and K denotes the symmedian point. Notice that O and H are not particularly nice in barycentric coordinates (as compared to in, say, complex numbers), but I and K are particularly elegant.

It is often more useful to convert the trigonometric forms of H and O into expressions entirely in terms of the side lengths by

$$O = (a^2 S_A : b^2 S_B : c^2 S_C)$$

and

$$H = (S_B S_C : S_C S_A : S_A S_B)$$

Table 7.1. *Barycentric Coordinates of the Centers of a Triangle.*

Point/Coordinates	Sketch of Proof
$G = (1 : 1 : 1)$	Trivial
$I = (a : b : c)$	Areal definition
$I_A = (-a : b : c)$, etc.	Areal definition
$K = (a^2 : b^2 : c^2)$	Isogonal conjugates
$H = (\tan A : \tan B : \tan C)$	Areal definition
$O = (\sin 2A : \sin 2B : \sin 2C)$	Areal definition

where we define

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}.$$

In Section 7.6 we investigate further properties of these expressions which provide a more viable way of dealing with them.

Just to provide some intuition on why Table 7.1 and Theorem 7.4 are useful, here is a simple example.

Example 7.7. Find the barycentric coordinates for the intersection of the internal angle bisector from A and the symmedian from B.

Solution. Suppose the desired intersection point is $P = (x : y : z)$. It is the intersection of lines AI and BK. According to Theorem 7.4, because $I = (a : b : c)$ we deduce that $y : z = b : c$. Similarly, because $K = (a^2 : b^2 : c^2)$ we deduce that $x : z = a^2 : c^2$. It is now elementary to find a solution to this: take

$$P = (a^2 : bc : c^2). \qquad \square$$

Moral: Cevians are extremely good in barycentric coordinates. And do not be afraid to use the law of sines if you have angles instead of side ratios.

Problems for this Section

Problem 7.8. Using the areal definition, show that $I = (a : b : c)$. Deduce the angle bisector theorem. **Hint**: 605

Problem 7.9. Find the barycentric coordinates for the intersection of the symmedian from A and the median from B. **Hint**: 463

7.3 Collinearity, Concurrence, and Points at Infinity

Theorem 7.1 can often be applied to show that three points are collinear. Specifically, we have the following result.

Theorem 7.10 (Collinearity). *Consider points P_1, P_2, P_3 with $P_i = (x_i : y_i : z_i)$ for $i = 1, 2, 3$. The three points are collinear if and only if*

$$0 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Note the coordinates need not be homogenized! This saves much computation.

Proof. The signed area of P_1, P_2, P_3 is zero (i.e., the points are collinear) if and only if

$$0 = \begin{vmatrix} \frac{x_1}{x_1+y_1+z_1} & \frac{y_1}{x_1+y_1+z_1} & \frac{z_1}{x_1+y_1+z_1} \\ \frac{x_2}{x_2+y_2+z_2} & \frac{y_2}{x_2+y_2+z_2} & \frac{z_2}{x_2+y_2+z_2} \\ \frac{x_3}{x_3+y_3+z_3} & \frac{y_3}{x_3+y_3+z_3} & \frac{z_3}{x_3+y_3+z_3} \end{vmatrix} \cdot [ABC].$$

The right-hand side simplifies as

$$\frac{[ABC]}{\prod_{i=1}^{3}(x_i + y_i + z_i)} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Because $[ABC] \neq 0$ the conclusion follows. □

This can be restated in the following useful form.

Proposition 7.11. *The line through two points $P = (x_1 : y_1 : z_1)$ and $Q = (x_2 : y_2 : z_2)$ is given precisely by the formula*

$$0 = \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

We often use this in combination with Theorem 7.4 in order to intersect a cevian with an arbitrary line through two points.

We also have a similar criterion for when three lines are concurrent. However, before proceeding, we make a remark about **points at infinity**. We earlier defined

$$(x : y : z) = \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z} \right)$$

whenever $x + y + z \neq 0$. What of the case $x + y + z = 0$?

Consider two parallel lines $u_1 x + v_1 y + w_1 z = 0$ and $u_2 x + v_2 y + w_2 z = 0$. Because they are parallel, we know that the system

$$\begin{aligned} 0 &= u_1 x + v_1 y + w_1 z \\ 0 &= u_2 x + v_2 y + w_2 z \\ 1 &= x + y + z \end{aligned}$$

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has no solutions (x, y, z). This is only possible when

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

However, this implies that the system of equations

$$\begin{aligned} 0 &= u_1x + v_1y + w_1z \\ 0 &= u_2x + v_2y + w_2z \\ 0 &= x + y + z \end{aligned}$$

has a nontrivial solution! (Conversely, if the lines are not parallel, the determinant is nonzero, and hence there is exactly one solution, namely $(0, 0, 0)$.)

In light of this, we make each of our lines just "a little longer" by adding one point to it, a *point at infinity*. It is a point $(x : y : z)$ satisfying the equation of the line and the additional condition $x + y + z = 0$. With this addition, every two lines intersect; the lines that were parallel before now correspond to lines that intersect at points at infinity. Points at infinity are defined more precisely at the start of Chapter 9.

Example 7.12. Find the point at infinity along the internal bisector of angle A.

Solution. The point at infinity is $(-(b + c) : b : c)$. After all, it lies on the equation of the angle bisector, and the sum of its coordinates is zero. □

Theorem 7.13 (Concurrence). *Consider three lines*

$$\ell_i : u_ix + v_iy + w_iz = 0$$

for $i = 1, 2, 3$. *They are concurrent or all parallel if and only if*

$$0 = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}.$$

Proof. This is essentially linear algebra. Consider the system of equations

$$\begin{aligned} 0 &= u_1x + v_1y + w_1z \\ 0 &= u_2x + v_2y + w_2z \\ 0 &= u_3x + v_3y + w_3z. \end{aligned}$$

It always has a solution $(x, y, z) = (0, 0, 0)$ and other solutions exist if and only if the lines concur (possibly at a point at infinity), which occurs only when the determinant of the matrix is zero. □

7.4 Displacement Vectors

In this section, we develop the notion of distance and direction through the use of vectors. This gives us a distance formula, and hence a circle formula, as well as a formula for the distance between two lines.

The chief definition is as follows. A **displacement vector** of two (normalized) points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ is denoted by $\overrightarrow{PQ}$ and is equal to $(q_1 - p_1, q_2 - p_2, q_3 - p_3)$. Note that the sum of the coordinates of a displacement vector is 0.

This section frequently involves translating the circumcenter O to the zero vector $\vec{0}$; this lets us invoke properties of the dot product described in Appendix A.3. This translation is valid since the point (x, y, z) satisfies $x + y + z = 1$, so the coordinates of the points do not change as a result; to be explicit, we can write

$$\vec{P} - \vec{O} = x\left(\vec{A} - \vec{O}\right) + y\left(\vec{B} - \vec{O}\right) + z\left(\vec{C} - \vec{O}\right)$$

since $x + y + z = 1$. As a result, however:

It is important that $x + y + z = 1$ when doing calculations with displacement vectors.

Our first major result is the distance formula.

Theorem 7.14 (Distance Formula). *Let P and Q be two arbitrary points and consider a displacement vector $\overrightarrow{PQ} = (x, y, z)$. Then the distance from P to Q is given by*

$$|PQ|^2 = -a^2yz - b^2zx - c^2xy.$$

Proof. Translate the coordinate plane so that the circumcenter O becomes the zero vector. Recall (from Appendix A.3) that this implies

$$\vec{A} \cdot \vec{A} = R^2 \text{ and } \vec{A} \cdot \vec{B} = R^2 - \frac{1}{2}c^2.$$

Here R is the circumradius of triangle ABC, as usual. Then we simply compute

$$|PQ|^2 = \left(x\vec{A} + y\vec{B} + z\vec{C}\right) \cdot \left(x\vec{A} + y\vec{B} + z\vec{C}\right).$$

Applying the properties of the dot product and using cyclic sum notation (defined in Section 0.3),

$$\begin{aligned}|PQ|^2 &= \sum_{\text{cyc}} x^2 \vec{A} \cdot \vec{A} + 2\sum_{\text{cyc}} xy\vec{A} \cdot \vec{B} \\ &= R^2(x^2 + y^2 + z^2) + 2\sum_{\text{cyc}} xy\left(R^2 - \frac{1}{2}c^2\right).\end{aligned}$$

Collecting the R^2 terms,

$$\begin{aligned}|PQ|^2 &= R^2(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) - (c^2xy + a^2yz + b^2zx) \\ &= R^2(x + y + z)^2 - a^2yz - b^2zx - c^2xy \\ &= -a^2yz - b^2zx - c^2xy\end{aligned}$$

since $x + y + z = 0$, being the sum of the coordinates in a displacement vector. $\square$

As a consequence we can deduce the formula for the equation of a circle. It looks unwieldy, but it can often be tamed; see the remarks that follow the proof.

Theorem 7.15 (Barycentric Circle). *The general equation of a circle is*

$$-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$$

for reals u, v, w.

Proof. Assume the circle has center (j, k, l) and radius r. Then applying the distance formula, we see that the circle is given by

$$-a^2(y - k)(z - l) - b^2(z - l)(x - j) - c^2(x - j)(y - k) = r^2.$$

Expand everything, and collect terms to get

$$-a^2yz - b^2zx - c^2xy + C_1x + C_2y + C_3z = C$$

for some hideous constants C_i and C. Since $x + y + z = 1$, we can rewrite

$$-a^2yz - b^2zx - c^2xy + ux + vy + wz = 0$$

as

$$-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$$

where $u = C_1 - C$, etc. $\square$

While this may look complicated, it turns out that circles that pass through vertices and sides are often very nice. For example, consider what occurs if the circle passes through $A = (1, 0, 0)$. The terms a^2yz, b^2zx, c^2xy all vanish, and accordingly we arrive at $u = 0$. Even if only one coordinate is zero, we still find many vanishing terms. Several examples are illustrated in the exercises.

As a result, whenever you are trying to solve a problem involving circumcircles through barycentrics, you should strive to set up the coordinates so that points on the circle are points on the sides, or better yet, vertices of the reference triangle. In other words, the choice of reference triangle is of paramount importance whenever circles appear.

Our last development for this section is a criterion to determine when two displacement vectors are perpendicular.

Theorem 7.16 (Barycentric Perpendiculars). *Let $\overrightarrow{MN} = (x_1, y_1, z_1)$ and $\overrightarrow{PQ} = (x_2, y_2, z_2)$ be displacement vectors. Then $\overline{MN} \perp \overline{PQ}$ if and only if*

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2).$$

The proof is essentially the same as before: shift $\vec{O}$ to the zero vector, and then expand the condition $\overrightarrow{MN} \cdot \overrightarrow{PQ} = 0$, which is equivalent to perpendicularity. We encourage you to prove the theorem yourself before reading the following proof.

Proof. Translate $\vec{O}$ to $\vec{0}$. It is necessary and sufficient that

$$\left(x_1\vec{A} + y_1\vec{B} + z_1\vec{C}\right) \cdot \left(x_2\vec{A} + y_2\vec{B} + z_2\vec{C}\right) = 0.$$

Expanding, this is just

$$\sum_{\text{cyc}} \left(x_1 x_2 \vec{A} \cdot \vec{A} \right) + \sum_{\text{cyc}} \left((x_1 y_2 + x_2 y_1) \, \vec{A} \cdot \vec{B} \right) = 0.$$

Taking advantage of the fact that $\vec{O} = 0$, we may rewrite this as

$$0 = \sum_{\text{cyc}} (x_1 x_2 R^2) + \sum_{\text{cyc}} (x_1 y_2 + x_2 y_1) \left(R^2 - \frac{c^2}{2} \right).$$

This rearranges as

$$\begin{aligned} R^2 \left(\sum_{\text{cyc}} (x_1 x_2) + \sum_{\text{cyc}} (x_1 y_2 + x_2 y_1) \right) &= \frac{1}{2} \sum_{\text{cyc}} \left((x_1 y_2 + x_2 y_1)(c^2) \right) \\ R^2 (x_1 + y_1 + z_1)(x_2 + y_2 + z_2) &= \frac{1}{2} \sum_{\text{cyc}} \left((x_1 y_2 + x_2 y_1)(c^2) \right). \end{aligned}$$

But we know that $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = 0$ in a displacement vector, so this becomes

$$\begin{aligned} R^2 \cdot 0 \cdot 0 &= \frac{1}{2} \sum_{\text{cyc}} \left((x_1 y_2 + x_2 y_1)(c^2) \right) \\ 0 &= \sum_{\text{cyc}} \left((x_1 y_2 + x_2 y_1)(c^2) \right). \end{aligned}$$ □

Theorem 7.16 is particularly useful when one of the displacement vectors is a side of the triangle. Several applications are given in the exercises, and more are seen in the examples section.

Problems for this Section

Lemma 7.17 (Barycentric Circumcircle). *The circumcircle (ABC) of the reference triangle has equation*

$$a^2 yz + b^2 zx + c^2 xy = 0.$$

Hint: 688

Problem 7.18. Consider a displacement vector $\overrightarrow{PQ} = (x_1, y_1, z_1)$. Show that $\overline{PQ} \perp \overline{BC}$ if and only if

$$0 = a^2(z_1 - y_1) + x_1(c^2 - b^2).$$

Lemma 7.19 (Barycentric Perpendicular Bisector). *The perpendicular bisector of $\overline{BC}$ has equation*

$$0 = a^2(z - y) + x(c^2 - b^2).$$

7.5 A Demonstration from the IMO Shortlist

Before proceeding to even more obscure theory, we take the time to discuss an illustrative example. Here is a problem from the IMO Shortlist of 2011.

Example 7.20 (Shortlist 2011/G6). Let ABC be a triangle with $AB = AC$ and let D be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C at the point E inside triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

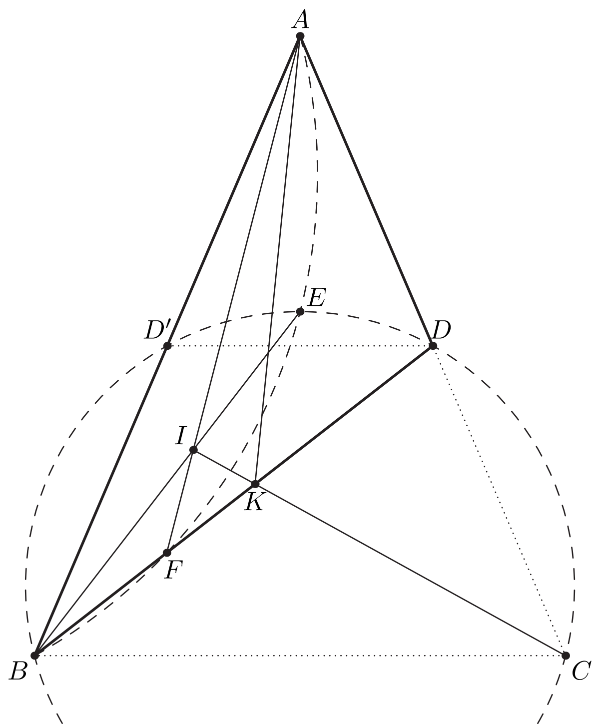

Figure 7.5A. IMO Shortlist 2011, Problem G6 (Example 7.20).

There are many nice and relatively painless synthetic observations that you can make in this problem. However, for the sake of discussion, we pretend we missed all of them. How should we apply barycentric coordinates?

Perhaps a better question is whether we should apply barycentric coordinates at all. There are two circles, but they seem relatively tame. There are lots of intersections of lines, but they seem to be mostly things that could be made into cevians. The final condition is about an angle bisector, which could pose difficulties, but we might make it.

A large part of this decision is based on what we choose for our reference triangle. At first we might be inclined to choose $\triangle ABC$, as the two circles in the problem pass through at least two vertices, and the condition $AB = AC$ is easy to encode. However, trying to prove that $\overline{BI}$ bisects $\angle ABD$, and that $\overline{AI}$ bisects $\angle BAK$, seems much less pleasant. Can we make at least one of them nicer?

That motivates a new choice of reference triangle: let us pick $\triangle ABD$ instead. That way, the $\overline{BE}$ bisection condition is extremely clean, and in fact almost immediate from the start (since E is the first point we compute). We still have the property that all circles pass through two vertices. Even better, the points F and K now lie on a side of the triangle, rather than just on some cevian (even though cevians are usually good too). And the second bisection condition looks much nicer now too, because we would only need to check $\frac{AB^2}{AK^2} = \frac{BF^2}{FK^2}$; since F and K lie on $\overline{BD}$, the right-hand side of this equality looks much better, and so the only truly nontrivial step would be computing AK^2. And finally, the isosceles condition is just $AB = 2AD$, which is trivial to encode.

It really is quite important that everything works out. A single thorn can doom the entire solution. We should always worry the most about the most time-consuming step of the entire plan—often this bottleneck takes longer to clear than the rest of the problem combined.

Let us begin. Set $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $D = (0, 0, 1)$, and denote $a = BD$, $b = AD$, $c = AB = 2b$. We also abbreviate $\angle A = \angle BAD$, $\angle B = \angle DBA$, and $\angle D = \angle ADB$.

Our first objective is to compute E, so we need the equation of (BDC). We know that C is the reflection of A over D, and hence $C = (-1, 0, 2)$. Thus we are plugging in $B = (0, 1, 0)$, $C = (-1, 0, 2)$, and $D = (0, 0, 1)$ into the circle equation

$$(BDC) : -a^2yz - b^2zx - c^2xy + (x + y + z)(ux + vy + wz) = 0.$$

The points B and D now force $v = w = 0$—indeed this is why we want circles to pass through vertices. Now plugging in C gives

$$2b^2 - u = 0 \Rightarrow u = 2b^2.$$

Great. Now E lies on the bisector of $\angle BAD$. Hence, set $E = (t : 1 : 2)$ (which is equivalent to $(bs : b : 2b) = (bs : b : c)$, where $s = \frac{t}{b}$) for some t. We can now solve for t by just dropping it into the circle equation, which gives

$$-a^2(1)(2) - b^2(2)(t) - c^2(t)(1) + (3 + t)(2b^2 \cdot t) = 0.$$

Putting $c = 2b$, we enjoy a cancellation of all the t terms, leaving us with merely $2b^2 \cdot t^2 = 2a^2$, and hence $t = \pm\frac{a}{b}$. We pick $t > 0$ since E is in the interior, and accordingly we deduce $E = \left(\frac{a}{b} : 1 : 2\right)$, or

$$E = (a : b : 2b) = (a : b : c).$$

This means E is the incenter of $\triangle ABD$! Glancing back at the diagram, that implies that $\overline{BE}$ is the angle bisector of $\angle ABD$. And the explanation is simple: if D' is the reflection of D across $\overline{AE}$, then the arcs $D'E$ and DE of (BCD) are equal by simple symmetry. Hence $\angle D'BE = \angle EBD$. Oops. That was embarrassing. But let us trudge on.

The next step is to compute the point F. We first need the equation of (AEB). By proceeding as before with generic u, v, w, we may derive that $u = v = 0$ with the points

A and B. As for E, we require

$$-a^2bc - b^2ca - c^2ab + (a+b+c)(cw) = 0 \Rightarrow w = ab.$$

Now set $F = (0 : m : n)$ and throw this into our discovered circle formula. The computations give us

$$-a^2mn + (m+n)(abn) = 0 \Rightarrow -am + b(m+n) = 0$$

and so $m : n = b : a - b$. Hence

$$F = (0 : b : a - b) = \left(0 : \frac{b}{a} : \frac{a-b}{a}\right).$$

Wait, that is pretty clean. Why might that be?

Upon further thought, we see that

$$DF = \frac{b}{a} \cdot BD = b = AD.$$

In other words, F is the reflection of A over the bisector $\overline{ED}$. Is this obvious? Yes, it is—the center of (AEB) lies on $\overline{ED}$ by our ubiquitous Lemma 1.18. Cue sound of slap against forehead.

(At this point we might take a moment to verify that $a > b$, to rule out configuration issues. This just follows from the triangle inequality $a + b > 2b$.)

Next, we compute I. This is trivial, because $\overline{AF}$ and $\overline{BE}$ are cevians. Verify that

$$I = (a(a-b) : bc : c(a-b)) = \left(a(a-b) : 2b^2 : 2b(a-b)\right)$$

is the correct point.

We now wish to compute K. Let us set $K = (0 : y : z)$ and solve again for $y : z$. Because the points I, K, and C are collinear, our collinearity criterion (Theorem 7.10) gives us

$$0 = \begin{vmatrix} 0 & y & z \\ -1 & 0 & 2 \\ a(a-b) & 2b^2 & 2b(a-b) \end{vmatrix}.$$

Let us see if we make more zeros. Add $a(a-b)$ times the second row to the last to obtain

$$0 = 2\begin{vmatrix} 0 & y & z \\ -1 & 0 & 2 \\ 0 & b^2 & (b+a)(a-b) \end{vmatrix}.$$

Here we have factored the naturally occurring 2 in the bottom row. Apparently this implies, upon evaluating by minors (in the first column) that we have

$$0 = \begin{vmatrix} y & z \\ b^2 & a^2 - b^2 \end{vmatrix}.$$

Hence we discover $K = \left(0 : b^2 : a^2 - b^2\right) = \left(0, \frac{b^2}{a^2}, \frac{a^2-b^2}{a^2}\right)$. This is really nice as well. Actually, it implies in a similar way as before that

$$DK = \frac{b^2}{a} = \frac{AD^2}{BD} \Rightarrow DB \cdot DK = AD^2.$$

Did we miss another synthetic observation? This new discovery implies $\triangle DAK \sim \triangle DBA$, and hence $\angle KAD = \angle KBA$. That would mean $\angle BAK = \angle A - \angle B$, which is positive by $a > b$.

Our calculations have given us $\angle BAK = \angle A - \angle B$, meaning it suffices to prove that $\angle BAF = \frac{1}{2}(\angle A - \angle B)$. And yet $\angle BAE = \frac{1}{2}\angle A$, so we only need to prove $\angle FAE = \frac{1}{2}\angle B$. In a blinding flash of obvious, $\angle FAE = \angle FBE = \frac{1}{2}\angle B$ and we are done.

The calculation of K from F encodes all of the nontrivial synthetic steps of the problem, and our surprise at the resulting K led us naturally to the end. We write this up nicely, hiding the fact that we ever missed such steps.

Solution to Example 7.20. Let D' be the midpoint of $\overline{AB}$. Evidently the points B, D', D, E, C are concyclic. By symmetry, $DE = D'E$, and hence $\overline{BE}$ is a bisector of $\angle D'BD$. It follows that E is the incenter of triangle ABD. Since the center of (AEB) lies on ray DE by Lemma 1.18, it follows that the reflection of A over $\overline{ED}$ lies on (AEB), and hence is F.

We now claim that $DK \cdot DB = DA^2$. The proof is by barycentric coordinates on $\triangle ABD$. Set $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$ and let $a = BD$, $b = AD$, and $c = AB = 2b$. The observations above imply that $F = (0 : b : b - a)$ and $E = (a : b : c)$. This implies

$$I = (a(a - b) : bc : c(a - b)) = \left(a(a - b) : 2b^2 : 2b(a - b)\right).$$

Finally, $C = (-1, 0, 2)$. Hence if $K = (0 : y : z)$ then we have

$$0 = \begin{vmatrix} 0 & y & z \\ -1 & 0 & 2 \\ a(a-b) & 2b^2 & 2b(a-b) \end{vmatrix} = \begin{vmatrix} 0 & y & z \\ -1 & 0 & 2 \\ 0 & 2b^2 & 2(a^2-b^2) \end{vmatrix}$$

so $y : z = b^2 : (a^2 - b^2)$, so $K = \left(0, \frac{b^2}{a^2}, 1 - \frac{b^2}{a^2}\right)$. It follows immediately that $DK = \frac{b^2}{a}$ as desired.

Now remark that

$$DK \cdot DB = DA^2 \Rightarrow \triangle DAK \sim \triangle DBA \Rightarrow \angle FAD = \angle B.$$

So $\angle BAK = \angle A - \angle B$. But $\angle EAD = \frac{1}{2}\angle A$ and $\angle FAE = \angle FBE = \frac{1}{2}\angle B$ imply $\angle BAF = \frac{1}{2}(\angle A - \angle B)$, and we are done. □

7.6 Conway's Notations

We now adapt **Conway's notation*** and define

$$S_A = \frac{b^2 + c^2 - a^2}{2}$$

and S_B and S_C analogously. Furthermore, let us define the shorthand $S_{BC} = S_B S_C$, and so on.

We first encountered these when we gave the coordinates of the circumcenter, and claimed they were friendlier than they seemed. This is because they happen to satisfy a

* The notation is named after John Horton Conway, a British mathematician.

lot of nice identities. For example, it is easy to see that $S_B + S_C = a^2$. Here are some less obvious ones.

Proposition 7.21 (Conway Identities). *Let S denote twice the area of triangle ABC. Then*

$$\begin{aligned} S^2 &= S_{AB} + S_{BC} + S_{CA} \\ &= S_{BC} + a^2 S_A \\ &= \frac{1}{2}(a^2 S_A + b^2 S_B + c^2 S_C) \\ &= (bc)^2 - S_A^2. \end{aligned}$$

In particular,

$$a^2 S_a + b^2 S_B - c^2 S_C = 2S_{AB}.$$

One might notice that there are a lot of $a^2 S_A$ and S_{AB} terms involved. This is because these are the coordinates of the circumcenter and orthocenter—hence these terms tend to arise naturally, and the identities provide a way of manipulating them.

More generally, if S is again equal to twice the area of triangle ABC, we define

$$S_\theta = S \cot\theta.$$

Here the angle is directed modulo 180°. The special case when $\theta = \angle A$ yields $S_A = \frac{1}{2}(b^2 + c^2 - a^2)$.

With this notation, we also have the following occasionally useful result.

Theorem 7.22 (Conway's Formula). *Let P be an arbitrary point. If $\beta = \measuredangle PBC$ and $\gamma = \measuredangle BCP$, then*

$$P = \left(-a^2 : S_C + S_\gamma : S_B + S_\beta\right).$$

The proof follows by computing the signed areas of triangles PBC, PAB, PCA and performing some manipulations. The proof is not particularly insightful and left to a diligent reader as an exercise. An example of an application appears in the exercises, Problem 7.37.

7.7 Displacement Vectors, Continued

In this section we refine some of our work in Section 7.4.

First of all, we look at our circle again:

$$-a^2yz - b^2zx - c^2xy + (x + y + z)(ux + vy + wz) = 0.$$

It might have seemed odd to insist on the negative signs in the first three terms, since we could have just as easily inverted the signs of u, v, w. It turns out that there is a good reason for this. Recall that we derived the circle formula by writing

$$(\text{distance from } (x, y, z) \text{ to center})^2 - \text{radius}^2 = 0.$$

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This should look familiar! What happens if we substitute an arbitrary point (x, y, z) into the formula? In that case we obtain the *power* of a point with respect to the circle. Explicitly, we obtain the following lemma.

Lemma 7.23 (Barycentric Power of a Point). *Let ω be the circle given by*

$$-a^2yz - b^2zx - c^2xy + (x + y + z)(ux + vy + wz) = 0.$$

Then let $P = (x, y, z)$ be any point. Then

$$\operatorname{Pow}_\omega(P) = -a^2yz - b^2zx - c^2xy + (x + y + z)(ux + vy + wz).$$

Note that we must have (x, y, z) homogenized here. Otherwise the distance formula breaks, and hence so does this lemma.

An easy but nonetheless indispensable consequence of Lemma 7.23 is the following lemma which gives us the radical axis of two circles.

Lemma 7.24 (Barycentric Radical Axis). *Suppose two non-concentric circles are given by the equations*

$$-a^2yz - b^2zx - c^2xy + (x + y + z)(u_1x + v_1y + w_1z) = 0$$
$$-a^2yz - b^2zx - c^2xy + (x + y + z)(u_2x + v_2y + w_2z) = 0.$$

Then their radical axis is given by

$$(u_1 - u_2)x + (v_1 - v_2)y + (w_1 - w_2)z = 0.$$

Proof. Just set the powers equal to each other and remark $x + y + z \neq 0$. Notice that this equation is homogeneous. □

We may also improve upon Theorem 7.16. In our proof of the theorem, we shifted $\vec{O}$ to zero and then used that

$$R^2(x_1 + y_1 + z_1)(x_2 + y_2 + z_2) = R^2 \cdot 0 \cdot 0 = 0.$$

In fact, we only need one of the displacement vectors to be zero for the entire product to be zero. For the other, we can get away with using a pseudo displacement vector; that is, we may cheat and, for example, write

$$\overrightarrow{HO} = \vec{H} - \vec{O} = \vec{H} = \vec{A} + \vec{B} + \vec{C} = (1, 1, 1)\,.$$

(Again, $\vec{O} = 0$ here. The lemma that $\vec{H} = \vec{A} + \vec{B} + \vec{C}$ under these conditions was proved in Chapter 6.)

Of course this is strictly nonsense, but the idea is there. Here is the formal statement.

Theorem 7.25 (Generalized Perpendicularity). *Suppose M, N, P, and Q are points with*

$$\overrightarrow{MN} = x_1\overrightarrow{AO} + y_1\overrightarrow{BO} + z_1\overrightarrow{CO}$$
$$\overrightarrow{PQ} = x_2\overrightarrow{AO} + y_2\overrightarrow{BO} + z_2\overrightarrow{CO}$$

such that either $x_1 + y_1 + z_1 = 0$ or $x_2 + y_2 + z_2 = 0$.

In that case, lines MN and PQ are perpendicular if and only if

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2).$$

Proof. Repeat the proof of Theorem 7.16. □

This becomes useful when O or H is involved in a perpendicularity. For example, we can obtain the following corollary by finding the perpendicular line to $\overline{AO}$ through A.

Example 7.26. The tangent to (ABC) at A is given by

$$b^2z + c^2y = 0.$$

Proof. Let $P = (x, y, z)$ be a point on the tangent and assume as usual that $\vec{O} = 0$. The displacement vector $\overrightarrow{PA}$ is

$$\overrightarrow{PA} = (x - 1, y, z) = (x - 1)\vec{A} + y\vec{B} + z\vec{C}.$$

We can also use the pseudo displacement vector

$$\overrightarrow{AO} = \vec{A} - \vec{O} = 1\vec{A} + 0\vec{B} + 0\vec{C}.$$

Putting $(x_1, y_1, z_1) = (x - 1, y, z)$ and $(x_2, y_2, z_2) = (1, 0, 0)$ yields the result. □

7.8 More Examples

Our first example is the famous Pascal's theorem from projective geometry.

Example 7.27 (Pascal's Theorem). Let A, B, C, D, E, F be six distinct points on a circle Γ. Prove that the three intersections of lines AB and DE, BC and EF, and CD and FA are collinear.

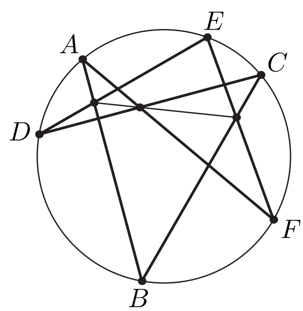

Figure 7.8A. Pascal's theorem (or one case thereof).

This problem seems okay because we have lots of intersections and only one circle.

Now we need to decide on a reference triangle. We might be tempted to pick ABC, but doing so loses much of the symmetry in the statement of Pascal's theorem. In addition, the lines DE and EF would fail to be cevians. Let us set reference triangle ACE instead—this way, our computations are symmetric, and the lines AB, DE, BC, EF, CD, FA are symmetric.

We can now proceed with the computation.

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Solution. In some terrible notation, let $a = CE$, $b = EA$, $c = AE$. Set $A = (1, 0, 0)$, $C = (0, 1, 0)$, $E = (0, 0, 1)$. We still have to deal with the other points, which have a lot of freedom. Now we write

$$B = (x_1 : y_1 : z_1)$$
$$D = (x_2 : y_2 : z_2)$$
$$F = (x_3 : y_3 : z_3)$$

and hope for the best. Here, the points are subject to the constraint that they must lie on (ACE). That is, we have that

$$-a^2 y_i z_i - b^2 z_i x_i - c^2 x_i y_i = 0, \qquad i = 1, 2, 3.$$

Hopefully this will be helpful later, but for now there is no clear way to use this.

Now to actually compute the intersections. First, we need to smash the cevians AB and ED together. (For organization, I am always writing the vertex of the reference triangle first.) The line AB is the locus of points $(x : y : z)$ with $y : z = y_1 : z_1$, while the line ED is the locus of points with $x : y = x_2 : y_2$. Hence, the intersection of lines AB and ED is

$$\overline{AB} \cap \overline{ED} = \left(\frac{x_2}{y_2} : 1 : \frac{z_1}{y_1} \right).$$

(Here we are borrowing the intersection notation from Chapter 9, a bit prematurely. Bear with me.) We can do the exact same procedure to determine the other intersections:

$$\overline{CD} \cap \overline{AF} = \left(\frac{x_2}{z_2} : \frac{y_3}{z_3} : 1 \right)$$

$$\overline{EF} \cap \overline{CB} = \left(1 : \frac{y_3}{x_3} : \frac{z_1}{x_1} \right).$$

Now to show that these are collinear, it suffices to show that the determinant

$$\begin{vmatrix} 1 & \frac{y_3}{x_3} & \frac{z_1}{x_1} \\ \frac{x_2}{y_2} & 1 & \frac{z_1}{y_1} \\ \frac{x_2}{z_2} & \frac{y_3}{z_3} & 1 \end{vmatrix}$$

is zero. (We have lined up the 1s on the main diagonal.) Seeing this, we are inspired to rewrite our given condition as

$$a^2 \cdot \frac{1}{x_1} + b^2 \cdot \frac{1}{y_1} + c^2 \cdot \frac{1}{z_1} = 0$$
$$a^2 \cdot \frac{1}{x_2} + b^2 \cdot \frac{1}{y_2} + c^2 \cdot \frac{1}{z_2} = 0$$
$$a^2 \cdot \frac{1}{x_3} + b^2 \cdot \frac{1}{y_3} + c^2 \cdot \frac{1}{z_3} = 0.$$

Linear algebra now tells us that

$$0 = \begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & \frac{1}{z_1} \\ \frac{1}{x_2} & \frac{1}{y_2} & \frac{1}{z_2} \\ \frac{1}{x_3} & \frac{1}{y_3} & \frac{1}{z_3} \end{vmatrix}$$

but this equals

$$\frac{1}{x_2 y_3 z_1} \cdot \begin{vmatrix} \frac{z_1}{x_1} & \frac{z_1}{y_1} & 1 \\ 1 & \frac{x_2}{y_2} & \frac{x_2}{z_2} \\ \frac{y_3}{x_3} & 1 & \frac{y_3}{z_3} \end{vmatrix}$$

which quickly implies that the first determinant is zero. □

There is actually little geometry involved in our proof of Pascal's theorem. In fact, there is very little special about the use of barycentric coordinates versus any other type of symmetric coordinates. Indeed they are a special case of **homogeneous coordinates**, i.e., a coordinate system that identifies $(kx : ky : kz)$ with (x, y, z). This is why the determinant calculations involved virtually no geometric observations.

Our next example involves a pair of incircles.

Example 7.28. Let ABC be a triangle and D a point on $\overline{BC}$. Let I_1 and I_2 denote the incenters of triangles ABD and ACD, respectively. Lines BI_2 and CI_1 meet at K. Prove that K lies on $\overline{AD}$ if and only if $\overline{AD}$ is the angle bisector of angle A.

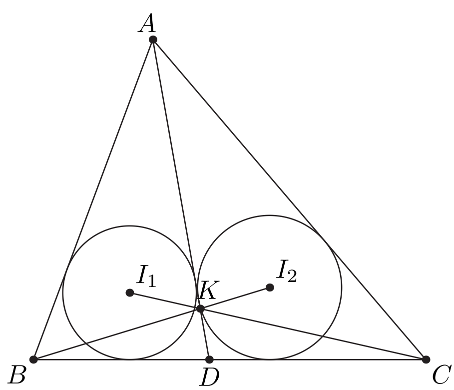

Figure 7.8B. Using barycentric coordinates to tame incircles.

The first thing we notice in this problem is the incenters. This should evoke fear, because we do not know much about how to deal with incenters other than that of ABC. Fortunately, these ones seem somewhat bound to ABC, so we might be okay.

We take ABC as the reference triangle. (After all, we do have a set of concurrent cevians, so this seems like something we want to use.) Now the hard part is deciding how to determine I_2.

Perhaps we can phrase I_2 as the intersection of two angle bisectors. Obviously one of them is the C-bisector. For the other, we consider the bisector $\overline{DI_2}$ (using $\overline{AI_2}$ will also work). If we can intersect the lines DI_2 and CI_2, this will of course give I_2.

So how can we handle $\overline{DI_2}$? If we let C_1 be the intersection of $\overline{DI_2}$ with $\overline{AC}$, then C_1 splits side $\overline{AC}$ in an $AD : AC$ ratio, by the angle bisector theorem. This suggests setting $d = AD$, $p = CD$, $q = BD$, where $p + q = a$. In that case, $C_1 = (p : 0 : d)$.

One might pause to worry about the fact we now have six variables. There are some relations, $p + q = a$ and Stewart's theorem, but we prefer not to use these. The reassurance is that so far all our equations have been of linear degree, so high degrees seem unlikely to appear. Indeed, we see that the solution is very short.

Solution to Example 7.28. Use barycentric coordinates with respect to ABC. Put $AD = d$, $CD = p$, $BD = q$.

Let ray DI_2 meet $\overline{AC}$ at C_1. Evidently $C_1 = (p : 0 : d)$ while $D = (0 : p : q)$.

Thus if $I_2 = (a : b : t)$ then we have

$$\begin{vmatrix} p & 0 & d \\ 0 & p & q \\ a & b & t \end{vmatrix} = 0 \Rightarrow t = \frac{ad + bq}{p}$$

which yields

$$I_2 = (ap : bp : ad + bq).$$

Similarly,

$$I_1 = (aq : ad + cp : cq).$$

So lines BI_2 and CI_1 intersect at a point

$$K = (apq : p(ad + cp) : q(ad + bq)).$$

This lies on line AD, so

$$\frac{p}{q} = \frac{p(ad + cp)}{q(ad + bq)}.$$

Hence we obtain $cp = bq$ or $p : q = b : c$ implying D is the foot of the angle bisector. $\square$

Next in line is a problem from the USAMO in 2008.

Example 7.29 (USAMO 2008/2). Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside triangle ABC. Prove that points A, N, F, and P all lie on one circle.

This one is actually a straightforward computation (but not a straightforward synthetic problem) with reference triangle ABC, but we have selected it to illustrate the use of

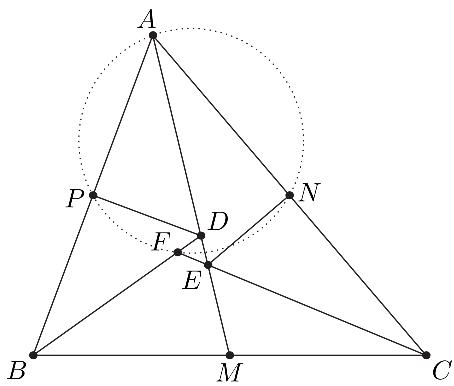

Figure 7.8C. Show that A, N, F, P are concyclic.

determinants and Conway's notation. There are only two nontrivial steps we will make. The first is to compute D as the intersection of lines PO and AM (where O is of course the circumcenter); there are other approaches but this is (I think) the cleanest. The second is that a homothety with ratio 2 at A to check that F lies on (ANP); we show that the reflection of A over F lies on (ABC), which solves the problem. All else is algebra.

Solution to Example 7.29. First, we find the coordinates of D. As D lies on $\overline{AM}$, we know $D = (t : 1 : 1)$ for some t. Now by Lemma 7.19, we find

$$0 = b^2(t-1) + (a^2 - c^2) \Rightarrow t = \frac{c^2 + b^2 - a^2}{b^2}.$$

Thus we obtain

$$D = \left(2S_A : c^2 : c^2\right).$$

Analogously $E = (2S_A : b^2 : b^2)$, and it follows that

$$F = \left(2S_A : b^2 : c^2\right).$$

The sum of the coordinates of F is

$$(b^2 + c^2 - a^2) + b^2 + c^2 = 2b^2 + 2c^2 - a^2.$$

Hence the reflection of A over F is simply

$$2F - A = \left(-a^2 : 2b^2 : 2c^2\right).$$

It is evident that F' lies on $(ABC) : -a^2yz - b^2zx - c^2xy = 0$, and we are done. $\square$

Our final example is the closing problem from Chapter 3. It stretches the power of our technique by showing even intersections with circles can be handled.

Example 7.30 (USA TSTST 2011/4). Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$.

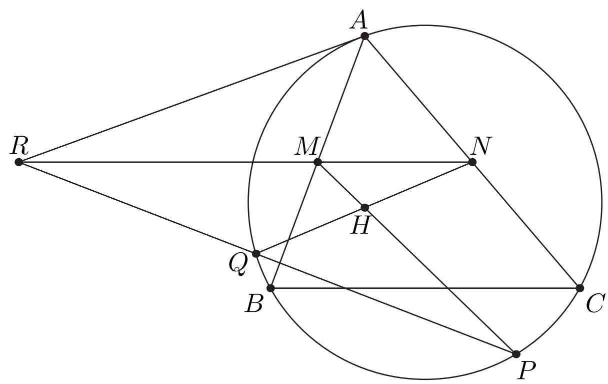

Figure 7.8D. Show that $\overline{RA}$ is a tangent.

This one is going to be wilder. We step back and plan before we begin the siege.

Intersecting $\overline{MN}$ and $\overline{PQ}$, and then showing the result is tangent, does not seem too hard. We have M, N, and H for free. However, it seems trickier to obtain the coordinates of P and Q.

Not all hope is lost. We want to avoid solving quadratics, so consider what happens when we intersect line MH with circle (ABC). Because $M = (1 : 1 : 0)$ and $H = (S_{BC} : S_{CA} : S_{AB})$, the equation of line MH can be computed as

$$0 = x - y + \left(\frac{S_{AC} - S_{BC}}{S_{AB}}\right) z.$$

Also, we of course know $0 = a^2yz + b^2zx + c^2xy$. Let us select $P = (x : y : -S_{AB})$. Then our system of equations in x and y is

$$\begin{aligned} x + y &= S_C(S_A - S_B) \\ c^2xy &= S_AS_B\left(a^2y + b^2x\right). \end{aligned}$$

We can attempt to solve directly for x, and we get some sloppy quadratic of the form $\alpha x^2 + \beta x + \gamma = 0$ for some (messy) expressions α, β, γ. The quadratic formula seems hopeless at this point.

But we are not stuck yet. Think about the two values of x. They correspond to the coordinates of two points, P and second point P', which has been marked in Figure 7.8E.

But the point P' is very familiar—it is just the point diametrically opposite C, and also the reflection of H over M. So it is straightforward to compute the value of x corresponding to P'. Vieta's formulas then tell us the sum of the roots of our quadratic is $-\frac{\beta}{\alpha}$, and we get our value of x for free.

Now we can start the computation.

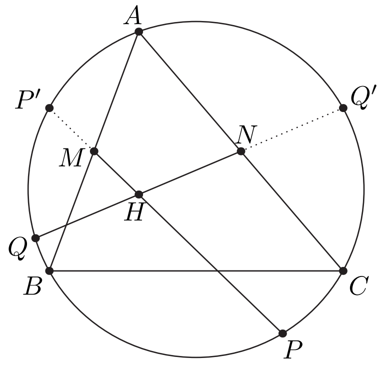

Figure 7.8E. Vieta jumping, anyone?

Solution to Example 7.30. We use barycentrics on ABC.

First, we compute the coordinates of P', the second intersection of line MH with (ABC). Since it is the reflection of $H = (S_{BC}, S_{CA}, S_{AB})$ over M, and the coordinates of H sum to $S_{AB} + S_{BC} + S_{CA}$, we may write

$$\begin{aligned} P' &= 2\left(\frac{S_{AB} + S_{BC} + S_{CA}}{2} : \frac{S_{AB} + S_{BC} + S_{CA}}{2} : 0\right) \\ &\quad - (S_{BC} : S_{CA} : S_{AB}) \\ &= (S_{AB} + S_{AC} : S_{AB} + S_{BC} : -S_{AB}) \\ &= \left(a^2 S_A : b^2 S_B : -S_{AB}\right). \end{aligned}$$

Now let us determine the coordinates of P, where we let $P = \left(x' : y' : z'\right) = \left(x' : y' : -S_{AB}\right)$ (valid since we just scale the coordinates so that $z' = -S_{AB}$). Because it lies on line MH, we find

$$0 = x' - y' + \left(\frac{S_{AC} - S_{BC}}{S_{AB}}\right) z' \Rightarrow y' = x' + S_{BC} - S_{AC}.$$

Also, we know that $a^2 y'z' + b^2 z'x' + c^2 x'y' = 0$, which gives

$$c^2 x'y' = S_{AB}\left(a^2 y' + b^2 x'\right).$$

Substituting, we have

$$c^2\left(x'\left(x' + S_{BC} - S_{AC}\right)\right) = S_{AB}\left(a^2\left(x' + S_{BC} - S_{AC}\right) + b^2 x'\right).$$

Collecting like terms gives the quadratic

$$c^2 x'^2 + \left[c^2\left(S_{BC} - S_{AC}\right) - (a^2 + b^2) S_{AB}\right] x' + \text{constant} = 0.$$

By Vieta's formulas, then, the x' we seek is just

$$\frac{a^2 + b^2}{c^2} S_{AB} - S_{BC} + S_{AC} - a^2 S_A.$$

Writing $a^2 = S_{AB} + S_{AC}$ in hopes of clearing out some terms, this becomes

$$\frac{a^2 + b^2 - c^2}{c^2} S_{AB} - S_{BC} = \frac{S_A S_B S_C}{c^2} - S_{BC}.$$

Now $y' = \frac{S_A S_B S_C}{c^2} - S_{AC}$. Cleaning further,

$$P = \left(S_B^2 S_C : S_A^2 S_C : c^2 S_{AB}\right).$$

Analogous calculations give that

$$Q = \left(S_B S_C^2 : b^2 S_{AC} : S_A^2 S_B\right).$$

Finding the equation of line PQ looks painful, so let us find where R should be first. Let the tangent to A meet line MN at R'. It is straightforward to derive that $R' = \left(b^2 - c^2 : b^2 : -c^2\right)$. Now we can just take a determinant. To show the three points P, Q, R' are collinear it suffices to check that

$$0 = \begin{vmatrix} S_B^2 S_C & S_A^2 S_C & c^2 S_A S_B \\ S_B S_C^2 & b^2 S_A S_C & S_A^2 S_B \\ b^2 - c^2 & b^2 & -c^2 \end{vmatrix}.$$

Note that $S_B^2 S_C - S_A^2 S_C - c^2 S_A S_B = c^2 \left[S_C(S_B - S_A) - S_A S_B\right]$. So upon subtracting the second and third columns from the first, this factors as

$$(S_{BC} - S_{AB} - S_{AC}) \cdot \begin{vmatrix} c^2 & S_A^2 S_C & c^2 S_A S_B \\ b^2 & b^2 S_A S_C & S_A^2 S_B \\ 0 & b^2 & -c^2 \end{vmatrix}.$$

To show this is zero, it suffices to check that

$$b^2 \left(c^2 S_A^2 S_B - b^2 c^2 S_A S_B\right) = c^2 \left(b^2 S_A^2 S_C - b^2 c^2 S_A S_C\right).$$

The left-hand side factors as $S_A S_B b^2 c^2 \left(S_A - b^2\right) = -S_A S_B S_C b^2 c^2$ and so does the right-hand side, so we are done. $\square$

This is certainly a somewhat brutal solution, but the calculation can be carried out within a half hour (and two pages) with some experience (and little insight). Notice how Conway's notation kept the expressions manageable.

7.9 When (Not) to Use Barycentric Coordinates

To summarize, let us discuss briefly when barycentrics are useful.

- Cevians are wonderful in every way, shape, and form. Know them, use them, love them. Pick reference triangles in which many lines become cevians.
- Problems heavily involving centers of a prominent triangle are in general good, because we have nice forms for most of the centers.
- Intersections of lines, collinearity, and concurrence are fine. Bonus points when cevians are involved.

- Problems that are symmetric around the vertices of a triangle. Because barycentric coordinates are also symmetric, this allows us to take advantage of the nice symmetry, unlike with Cartesian coordinates.
- Ratios, lengths, or areas.
- Problems with few points. This is kind of obvious—the fewer points you have to compute, the better.

In contrast, here are things that barycentric coordinates do not handle well.

- Lots of circles. One can sometimes find a way around circles (for example, if only the radical axis or power of a point is relevant).
- Circles that do not pass through vertices of sides of a reference triangle. In general, the equation of a circle through three completely arbitrary points will be very ugly. However, the circle becomes much more tractable if the points it passes through have zeros.
- Arbitrary circumcenters.
- General angle conditions. Of course, there are exceptions; they typically involve angle conditions that can be translated into length conditions. The angle bisector theorem is your friend here.

7.10 Problems

There are quite a few contest problems that can be solved by barycentrics; this represents a rather small subset of problems I have encountered that are susceptible. Part of the reason is that, at the time of writing, barycentrics are a relatively unknown technique. As a result, test-writers are not aware when a problem they propose is trivialized by barycentric coordinates, as they would have been for a problem approachable by either complex numbers or Cartesian coordinates.

Lemma 7.31. *Let ABC be a triangle with altitude $\overline{AL}$ and let M be the midpoint of $\overline{AL}$. If K is the symmedian point of triangle ABC, prove that $\overline{KM}$ bisects $\overline{BC}$.* **Hints**: 652 393

Problem 7.32. Let I and G denote the incenter and centroid of a triangle ABC and let N denote the **Nagel point**; this is the intersection of the cevians that join A to the contact point of the A-excircle on $\overline{BC}$, and similarly for B and C. Prove that I, G, N are collinear and that $NG = 2GI$. **Hints**: 271 243

Problem 7.33 (IMO 2014/4). Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC. **Hints**: 486 574 251 **Sol:** p.265

Problem 7.34 (EGMO 2013/1). The side BC of triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then triangle ABC is right-angled. **Hint**: 188 **Sol:** p.265

Problem 7.35 (ELMO Shortlist 2013). In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets $\overline{AC}$ at F (other than A), and the circumcircle of ADC meets

$\overline{AB}$ at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC. **Hints**: 657 653

Problem 7.36 (IMO 2012/1). Given triangle ABC the point J is the center of the excircle opposite the vertex A. This excircle is tangent to side BC at M, and to lines AB and AC at K and L, respectively. Lines LM and BJ meet at F, and lines KM and CJ meet at G. Let S be the point of intersection of lines AF and BC, and let T be the point of intersection of lines AG and BC. Prove that M is the midpoint of ST. **Hints**: 447 280 **Sol:** p.266

Problem 7.37 (Shortlist 2001/G1). Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side $\overline{BC}$. Thus one of the two remaining vertices of the square is on side $\overline{AB}$ and the other is on $\overline{AC}$. Points B_1, C_1 are defined in a similar way for inscribed squares with two vertices on sides $\overline{AC}$ and $\overline{AB}$, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent. **Hints**: 123 466

Problem 7.38 (USA TST 2008/7). Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $\overline{PQ} \parallel \overline{AB}$ and $\overline{PR} \parallel \overline{AC}$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$. **Hints**: 6 647 **Sol:** p.266

Problem 7.39 (USAMO 2001/2). Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$. **Hints**: 320 160

Problem 7.40 (USA TSTST 2012/7). Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN. **Hints**: 381 345 576

Problem 7.41. Let ABC be a triangle with incenter I. Let P and Q denote the reflections of B and C across $\overline{CI}$ and $\overline{BI}$, respectively. Show that $\overline{PQ} \perp \overline{OI}$, where O is the circumcenter of ABC. **Hints**: 396 461

Lemma 7.42. *Let ABC be a triangle with circumcircle Ω and let T_A denote the tangency points of the A-mixtilinear incircle to Ω. Define T_B and T_C similarly. Prove that lines AT_A, BT_B, CT_C, IO are concurrent, where I and O denote the incenter and circumcenter of triangle ABC.* **Hints**: 490 54 602 488 **Sol:** p.267

Problem 7.43 (USA December TST for IMO 2012). In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side

BC, the circumcircle of triangle ADE passes through a fixed point other than A. **Hints**: 179 144 137

Problem 7.44 (Sharygin 2013). Let C_1 be an arbitrary point on side AB of $\triangle ABC$. Points A_1 and B_1 are on rays BC and AC such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point. **Hints**: 51 12 66 304 **Sol:** p.268

Problem 7.45 (APMO 2013/5). Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that $\overline{PB}$ and $\overline{PD}$ are tangent to ω. The tangent at C intersects $\overline{PD}$ at Q and the line AD at R. Let E be the second point of intersection between $\overline{AQ}$ and ω. Prove that B, E, R are collinear. **Hints**: 379 524 129

Problem 7.46 (USAMO 2005/3). Let ABC be an acute-angled triangle, and let P and Q be two points on its side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $\overline{QC_1} \parallel \overline{CA}$, and C_1 and Q lie on opposite sides of line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $\overline{QB_1} \parallel \overline{BA}$, and B_1 and Q lie on opposite sides of line AC. Prove that the points B_1, C_1, P, and Q lie on a circle. **Hints**: 191 325 204

Problem 7.47 (Shortlist 2011/G2). Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. For $1 \le i \le 4$, let O_i and r_i be the circumcenter and the circumradius of triangle $A_{i+1}A_{i+2}A_{i+3}$ (where $A_{i+4} = A_i$). Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

Hints: 468 588 224 621 **Sol:** p.269

Problem 7.48 (Romania TST 2010). Let ABC be a scalene triangle, let I be its incenter, and let A_1, B_1, and C_1 be the points of contact of the excircles with the sides BC, CA, and AB, respectively. Prove that the circumcircles of the triangles AIA_1, BIB_1, and CIC_1 have a common point different from I. **Hints**: 549 23 94

Problem 7.49 (ELMO 2012/5). Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $\overline{PD} \parallel \overline{AE}$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$. **Hints**: 171 229 **Sol:** p.270

Problem 7.50 (USA TST 2004/4). Let ABC be a triangle. Choose a point D in its interior. Let ω_1 be a circle passing through B and D and ω_2 be a circle passing through C and D so that the other point of intersection of the two circles lies on $\overline{AD}$. Let ω_1 and ω_2 intersect side BC at E and F, respectively. Denote by X the intersection of lines DF and AB, and let Y the intersection of DE and AC. Show that $\overline{XY} \parallel \overline{BC}$. **Hints**: 301 206 567 126

Problem 7.51 (USA TSTST 2012/2). Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and the circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let

M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $\overline{MN} \parallel \overline{O_1O_2}$. **Hints**: 651 518 664 364

Problem 7.52 (IMO 2004/5). In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. Point P lies inside $ABCD$ with $\angle PCB = \angle DBA$ and $\angle PDC = \angle BDA$. Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$. **Hints**: 117 266 641 349 **Sol:** p.270

Problem 7.53 (Shortlist 2006/G4). Let ABC be a triangle with $\angle C < \angle A < 90^\circ$. Select point D on side AC so that $BD = BA$. The incircle of ABC is tangent to $\overline{AB}$ and $\overline{AC}$ at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that the line KL bisects $\overline{AJ}$. **Hints**: 5 295 281 394