CHAPTER 9

Projective Geometry

Projective geometry is all geometry. Arthur Cayley

In the previous chapter we studied inversion, a transformation that deals with circles. It also happened to nicely preserve incidence, i.e., inversion preserves intersections. Projective geometry features a powerful set of tools that this time focus primarily on analyzing incidence. Problems that mostly deal with intersections, parallel lines, tangent circles, and so on, often succumb to projective geometry.

9.1 Completing the Plane

First, we set up the projective plane with points at infinity.

Imagine we are walking down the infinitely long corridor in Figure 9.1A and take a moment to look around us.

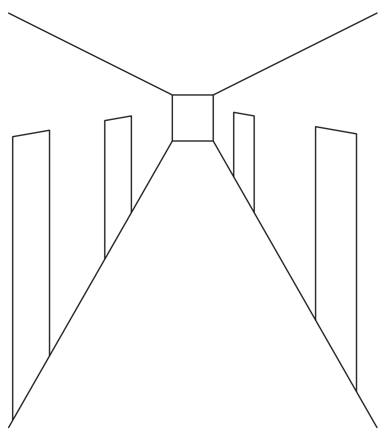

Figure 9.1A. A long hallway with a few doors.

There are some parallel lines in the figure, say the two lines that mark the floor. But they are not actually parallel in the picture: the two lines are converging towards a point. In fact, all the parallel lines are converging towards the same point on the horizon. So it does seem like parallel lines intersect infinitely far away, even in a plane (for example, consider the left wall or the right wall).

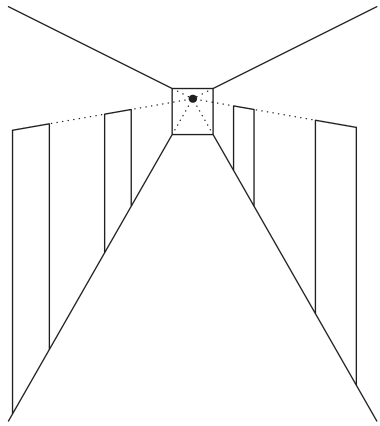

Figure 9.1B. Are the parallel lines really parallel?

The **real projective plane** uses precisely this idea. In addition to the standard points of Euclidean plane (which we call **Euclidean points**), it also includes a **point at infinity** for each class of parallel lines (one can think of this as adding a point at infinity for each direction). To be more precise, we partition all the lines of the Euclidean plane into equivalence classes (called **pencils** of parallel lines) where two distinct lines are in the same class if they are parallel. Then we add a point at infinity for each pencil. We also add one extra line, the **line at infinity**, comprising exactly of all the points at infinity.

With this modification, any two lines do in fact intersect at exactly one point. The intersection of two non-parallel lines is a Euclidean point, while two parallel lines meet at the point at infinity. The use of this convention lets us replace the clumsy language of "concurrent or all parallel" (as in Theorem 2.9).

Finally, throughout this chapter we use a special shorthand. For points A, B, C, D, let $\overline{AB} \cap \overline{CD}$ denote the intersection of lines AB and CD, possibly at infinity.

9.2 Cross Ratios

The **cross ratio** is an important invariant in projective geometry. Given four collinear points A, B, X, Y (which may be points at infinity), we define the cross ratio as

$$(A, B; X, Y) = \frac{XA}{XB} \div \frac{YA}{YB}.$$

Here the ratios are directed with the same convention as Menelaus's theorem; in particular, the cross ratio can be negative! If A, B, X, Y lie on a number line then this can be written as

$$(A, B; X, Y) = \frac{x-a}{x-b} \div \frac{y-a}{y-b}.$$

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You can check that $(A, B; X, Y) > 0$ precisely when segments $\overline{AB}$ and $\overline{XY}$ are disjoint or one is contained inside the other. We also generally assume $A \neq X$, $B \neq X$, $A \neq Y$, $B \neq Y$.

We can also define the cross ratio for four lines a, b, x, y concurrent at some point P. If $\angle(\ell, m)$ is the angle between the two lines ℓ and m, then we can write

$$(a, b; x, y) = \pm \frac{\sin \angle(x, a)}{\sin \angle(x, b)} \div \frac{\sin \angle(y, a)}{\sin \angle(y, b)}.$$

The sign is chosen in a similar manner as the procedure for four points: if one of the four angles formed by line a and b contains neither x nor y, then $(a, b; x, y)$ is positive; otherwise it is negative.

If A, B, X, Y are collinear points on lines a, b, x, y (respectively) concurrent at P, we write

$$P(A, B; X, Y) = (a, b; x, y).$$

The structure $P(A, B; X, Y)$ is called a **pencil** of lines. See Figure 9.2A.

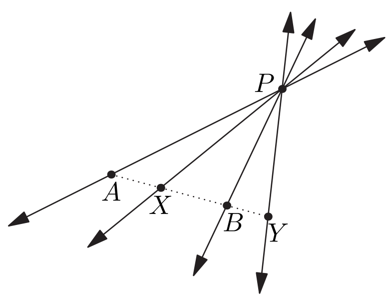

Figure 9.2A. Actually, $P(A, B; X, Y) = (A, B; X, Y)$.

As you might have already guessed, the sign convention for the trigonometric form is just contrived so that the following theorem holds.

Theorem 9.1 (Cross-Ratio Under Perspectivity). *Suppose that $P(A, B; X, Y)$ is a pencil of lines. If A, B, X, Y are collinear then*

$$P(A, B; X, Y) = (A, B; X, Y).$$

Proof. This is just a computation with the law of sines on $\triangle XPA$, $\triangle XPB$, $\triangle YPA$, $\triangle YPB$. There are multiple configurations to check, but they are not so different. $\square$

We can even define the cross ratio for four points on a circle, as follows:

Theorem 9.2 (Cross Ratios on Cyclic Quadrilaterals). *Let A, B, X, Y be concyclic. If P is any point on its circumcircle, then $P(A, B; X, Y)$ does not depend on P. Moreover,*

$$P(A, B; X, Y) = \pm \frac{XA}{XB} \div \frac{YA}{YB}$$

where the sign is positive if $\overline{AB}$ and $\overline{XY}$ do not intersect, and negative otherwise.

The invariance just follows from the fact that the angles are preserved as P varies around the circle. Hence, we just define the cross ratio of four concyclic points to be the value of $P(A, B; X, Y)$ for any particular P. The actual ratio $\frac{XA}{XB} : \frac{YA}{YB}$ follows by applying the law of sines and the details are left as an exercise.

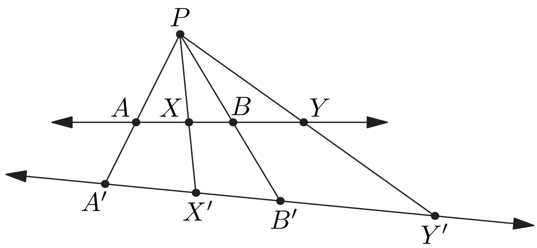

Figure 9.2B. Taking perspectivity at P.

Why do we care? Consider the situation in Figure 9.2B. Two lines ℓ and m are given, and points A, B, X, Y are on ℓ. We can pick any point P and consider the intersections of lines PA, PB, PX, PY with m, say A', B', X', Y'. Then

$$(A, B; X, Y) = P(A, B; X, Y) = P(A', B'; X', Y') = (A', B'; X', Y').$$

In effect, that means we have the power to project $(A, B; X, Y)$ from line ℓ onto line m. This is called taking **perspectivity** at P. We often denote this by

$$(A, B; X, Y) \stackrel{P}{=} (A', B'; X', Y').$$

The same technique can be done if P, A, X, B, Y are concyclic, in which case we may project onto a line. Conversely, given $(A, B; X, Y)$ on a line we may pull from P onto circle through P, as in Figure 9.2C (and vice versa). The important thing is that these operations all preserve the cross ratio $(A, B; X, Y)$.

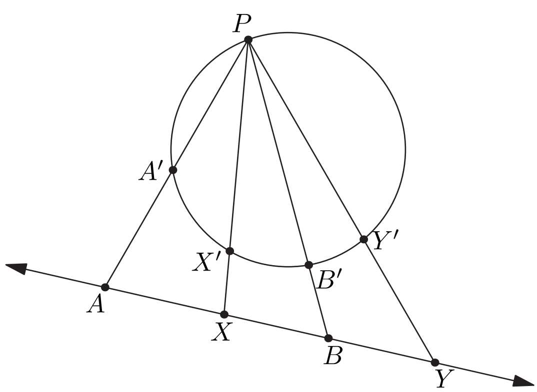

Figure 9.2C. Projecting via P from a line onto a circle through P.

The fact that cross ratio is preserved under all of these is why it is well-suited for problems that deal with lots of intersections. One can even think of chasing cross ratios around the diagram by repeatedly applying perspectives. We see more of this in later examples.

In the next section we investigate the most important case of the cross ratio, the harmonic bundle.

Problems for this Section

Problem 9.3. Check that

$$(A, B; X, Y) = (B, A; X, Y)^{-1} = (A, B; Y, X)^{-1} = (X, Y; A, B).$$

for any four distinct points A, B, X, Y.

Problem 9.4. Let A, B, X be distinct collinear points and k a real number. Prove that there is exactly one point Y (possibly the point at infinity) such that $(A, B; X, Y) = k$. **Hint**: 287

Problem 9.5. In Figure 9.2A, is $P(A, B; X, Y)$ positive or negative? **Hint**: 83

Problem 9.6. Let A, B, X be collinear points and P_∞ a point at infinity along their common line. What is $(A, B; X, P_\infty)$? **Hint**: 666

Problem 9.7. Give the proof of Theorem 9.2.

9.3 Harmonic Bundles

The most important case of our cross ratio is when $(A, B; X, Y) = -1$. We say that $(A, B; X, Y)$ is a **harmonic bundle** in this case, or just **harmonic**. Furthermore, a cyclic quadrilateral $AXBY$ is a **harmonic quadrilateral** if $(A, B; X, Y) = -1$.

Observe that if $(A, B; X, Y) = -1$, then $(A, B; Y, X) = (B, A; X, Y) = -1$. We sometimes also say that Y is the **harmonic conjugate** of X with respect to $\overline{AB}$; as the name suggests, it is unique, and the harmonic conjugate of Y is X itself.

Harmonic bundles are important because they appear naturally in many configurations. We present four configurations in which they arise.

The first lemma is trivial to prove, but gives us a new way to handle midpoints, particularly if they appear along with parallel lines.

Lemma 9.8 (Midpoints and Parallel Lines). *Given points A and B, let M be the midpoint of $\overline{AB}$ and P_∞ the point at infinity of line AB. Then $(A, B; M, P_\infty)$ is a harmonic bundle.*

The next lemma (illustrated in Figure 9.3A) describes harmonic quadrilaterals in terms of tangents to a circle.

Lemma 9.9 (Harmonic Quadrilaterals). *Let ω be a circle and let P be a point outside it. Let $\overline{PX}$ and $\overline{PY}$ be tangents to ω. Take a line through P intersecting ω again at A and B. Then*

(a) $AXBY$ is a harmonic quadrilateral.
(b) If $Q = \overline{AB} \cap \overline{XY}$, then $(A, B; Q, P)$ is a harmonic bundle.

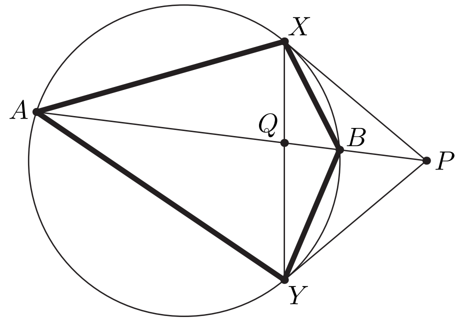

Figure 9.3A. A harmonic quadrilateral. $(A, B; P, Q)$ is also harmonic.

Proof. We use symmedians. We obtain $\frac{XA}{XB} = \frac{YA}{YB}$ from Lemma 4.26, and $(A, B; X, Y)$ is negative by construction. This establishes that $AXBY$ is harmonic.

To see that $(A, B; Q, P)$ is harmonic, just write

$$(A, B; X, Y) \stackrel{X}{=} (A, B; Q, P).$$

Here we are projecting from the circle onto the line AB from X, noting that line XX in this context is actually just the tangent to ω. (To see this, consider the behavior of line XX' when X' is very close to X on the circle.) □

This also implies the tangents to A and B intersect on line XY. (Why?)

An important special case is when $\overline{AB}$ is selected as a diameter of ω. In that case, P and Q are inverses when inverting around ω. In full detail, we have the following.

Proposition 9.10 (Inversion Induces Harmonic Bundles). *Let P be a point on line $\overline{AB}$, and let P^* denote the image of P after inverting around the circle with diameter $\overline{AB}$. Then $(A, B; P, P^*)$ is harmonic.*

The third and fourth lemmas involve no circles at all. Actually the fourth is really a consequence of the third.

Lemma 9.11 (Cevians Induces Harmonic Bundles). *Let ABC be a triangle with concurrent cevians $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ (possibly on the extensions of the sides). Line EF meets BC at X (possibly at a point at infinity). Then $(X, D; B, C)$ is a harmonic bundle.*

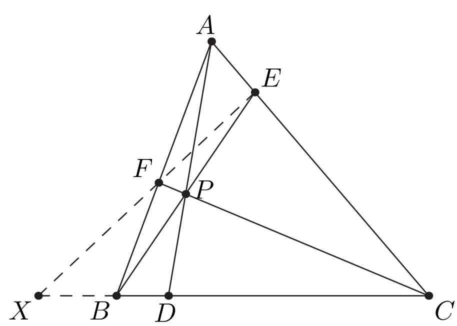

Figure 9.3B. Ceva's and Menelaus's theorems produce $(X, D; B, C) = -1$.

Proof. Use the directed form Ceva's theorem and Menelaus's theorem on Figure 9.3B. $\square$

Lemma 9.12 (Complete Quadrilaterals Induces Harmonic Bundles). *Let $ABCD$ be a quadrilateral whose diagonals meet at K. Lines AD and BC meet at L, and line KL meets $\overline{AB}$ and $\overline{CD}$ at M and N. Then $(K, L; M, N)$ is a harmonic bundle.*

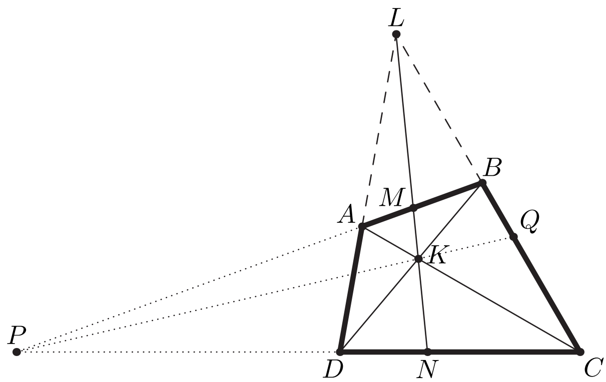

Figure 9.3C. You can modify Lemma 9.11 to get $(K, L; M, N)$ a harmonic bundle as well.

Proof. As in Figure 9.3C, let $P = \overline{AB} \cap \overline{CD}$, and let $Q = \overline{PK} \cap \overline{BC}$. By Lemma 9.11, $(Q, L; B, C) = -1$. Projecting onto the desired line, we derive

$$-1 = (Q, L; B, C) \stackrel{P}{=} (K, L; M, N). \qquad \square$$

Harmonic bundles let us move from one of these configurations to the others. As an example, we revisit Problem 4.45.

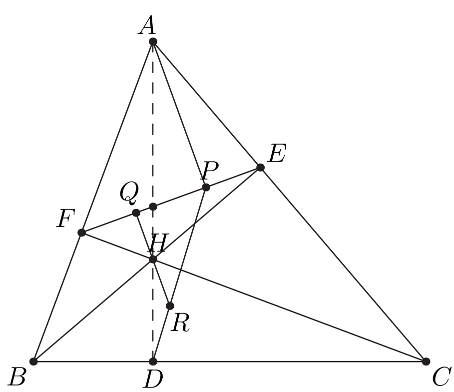

Figure 9.3D. The first problem from the USA TST 2011.

Example 9.13 (USA TST 2011/1). In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{BE} \perp \overline{CA}$, $\overline{CF} \perp \overline{AB}$.

Altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ meet at orthocenter H. Points P and Q lie on segment $\overline{EF}$ such that $\overline{AP} \perp \overline{EF}$ and $\overline{HQ} \perp \overline{EF}$. Lines DP and QH intersect at point R. Compute HQ/HR.

We might readily dismiss this as an uninteresting problem. The answer is 1; the problem is just Lemma 4.9 applied to triangle DEF. However, it turns out there is a quick projective proof completely independent of this.

Remember Lemma 9.8? We indeed have both a midpoint (H of $\overline{QR}$) and a line parallel to it ($\overline{AP} \parallel \overline{QR}$). Hence we take perspectivity through P. More precisely, let P_∞ be the point at infinity for $\overline{AP}$ and $\overline{QR}$. Then

$$(Q, R; H, P_\infty) \stackrel{P}{=} (\overline{QP} \cap \overline{AD}, D; H, A).$$

If we can show the latter is a harmonic bundle, then we are done. But this is just Lemma 9.12!

Needless to say, we can go backwards, as in the proof below.

Solution. By Lemma 9.12, $(A, H; \overline{AD} \cap \overline{EF}, D) = -1$. Projecting through P, we find $(P_\infty, H; Q, R) = -1$, where P_∞ is the point at infinity on parallel lines AP and QR. Hence $\frac{HQ}{HR} = 1$. □

Problems for this Section

Problem 9.14. Check the details in the proofs of Lemma 9.11 and Lemma 9.18.

Problem 9.15. In the coordinate plane, the points $A = (-1, 0)$, $B = (1, 0)$, $X = \left(\frac{1}{100}, 0\right)$ and $Y = (m, 0)$ form a harmonic bundle $(A, B; X, Y) = -1$. What is m? **Hint**: 334

Problem 9.16. Show that Problem 1.43 (see Figure 9.3E) is immediate from the tools developed in this chapter. **Hints**: 107 687 607 451 520

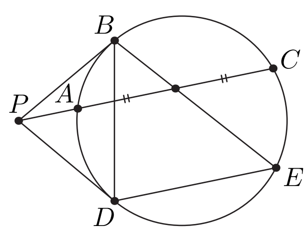

Figure 9.3E. Solve JMO 2011/5 (Problem 1.43) using harmonic bundles.

Lemma 9.17 (Midpoint Lengths). *Points A, X, B, P lie on a line in that order, and $(A, B; X, P) = -1$. Let M be the midpoint of $\overline{AB}$. Show that $MX \cdot MP = \left(\frac{1}{2}AB\right)^2$ and $PX \cdot PM = PA \cdot PB$.* **Hints**: 41 557

9.4 Apollonian Circles

There is one additional configuration with naturally occurring harmonic bundles. First, we need to state a lemma (see Figure 9.4A).

Lemma 9.18 (Right Angles and Bisectors). *Let X, A, Y, B be collinear points in that order and let C be any point not on this line. Then any two of the following conditions implies the third condition.*

(i) $(A, B; X, Y)$ *is a harmonic bundle.*
(ii) $\angle XCY = 90^\circ$.
(iii) $\overline{CY}$ *bisects* $\angle ACB$.

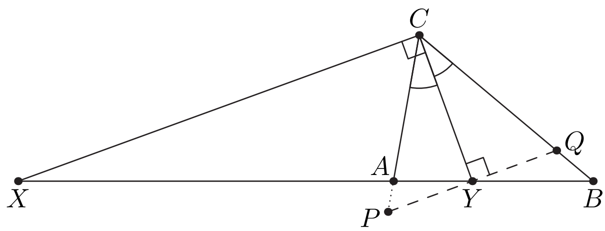

Figure 9.4A. $\overline{CX}$ and $\overline{CY}$ are external and internal angle bisectors.

Proof. There is a straightforward trigonometric proof, but here we present a synthetic solution. Draw the line through Y parallel to $\overline{CX}$ and let it intersect rays CA and CB at P and Q, respectively. Since $\triangle XAC \sim \triangle YAP$ and $\triangle XBC \sim \triangle YBQ$, we derive

$$PY = \frac{AY}{AX} \cdot CX \text{ and } QY = \frac{BY}{BX} \cdot CX.$$

Thus $PY = QY$ if and only if $(A, B; X, Y) = -1$. Now any two of the conditions imply $\triangle CYP$ and $\triangle CYQ$ are congruent, which gives the third. □

While this is useful in its own right, it leads directly to the so-called **Apollonian circle**, which is a way of linking angles with ratios. The statement is as follows.

Theorem 9.19 (Apollonian Circles). *Let $\overline{AB}$ be a segment and $k \neq 1$ be a positive real. The locus of points C satisfying $\frac{CA}{CB} = k$ is a circle whose diameter lies on $\overline{AB}$.*

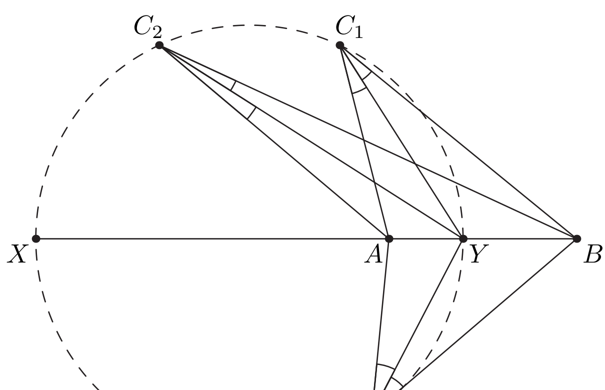

Figure 9.4B. Apollonian Circles

This is really just a restatement of Lemma 9.18, with the congruent angles rewritten as a ratio because of the angle bisector theorem. Here are the details; refer to Figure 9.4B.

Proof. First of all, let X and Y be the two points on line AB with

$$\frac{XA}{XB} = \frac{YA}{YB} = k.$$

Without loss of generality, Y lies on $\overline{AB}$.

Now observe that for any other point C, $\frac{CA}{CB} = k$ is just equivalent to $\angle CAY = \angle YBC$ by the angle bisector theorem. That is equivalent to $\angle XCY = 90^\circ$ by Lemma 9.18, and hence we discover the Apollonian circle. ☐

Problems for this Section

Problem 9.20. In the notation of Figure 9.4B, what is the Apollonian circle of $\overline{XY}$ through A? **Hints**: 411 70

Problem 9.21. Check that as k varies, the resulting set of circles are all coaxial*. **Hints**: 315 147

Lemma 9.22 (Harmonic Bundles on the Bisector). *Let ABC be a triangle with incenter I and A-excenter I_A. Prove that*

$$(I, I_A; A, \overline{AI} \cap \overline{BC}) = -1.$$

9.5 Poles/Polars and Brocard's Theorem

Projective and inversive techniques are actually closely related by the concepts of poles and polars.

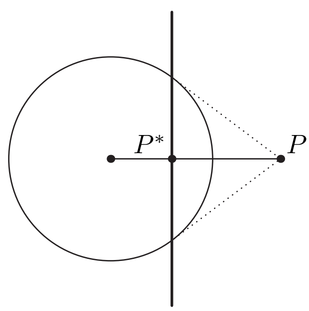

Figure 9.5A. The polar of point P is the line shown.

Fix a circle ω with center O and a point P. Let P^* be the inverse of P with respect to inversion around ω. The **polar** of point P (possibly at infinity and distinct from O) is the line passing through P^* perpendicular to $\overline{OP}$. As we have mentioned before, when P is outside circle ω then its polar is the line through the two tangency points from P to ω. The polar of O is just the line at infinity.

* Actually, it turns out any non-intersecting coaxial circles are Apollonian.

Similarly, given a line ℓ not through O, we define its **pole**† as the point P that has ℓ as its polar.

First, an obvious result that is nonetheless useful.

Theorem 9.23 (La Hire's Theorem). *A point X lies on the polar of a point Y if and only if Y lies on the polar of X.*

Proof. Left as an exercise. It is merely similar triangles. □

La Hire's theorem demonstrates a concept called **duality**: one can exchange points for lines, lines for intersections, collinearity for concurrence. Simply swap every point with its polar and every line with its pole.

We can now state an important result relating poles and polars to harmonic bundles.

Proposition 9.24. *Let $\overline{AB}$ be a chord of a circle ω and select points P and Q on line AB. Then $(A, B; P, Q) = -1$ if and only if P lies on the polar of Q.*

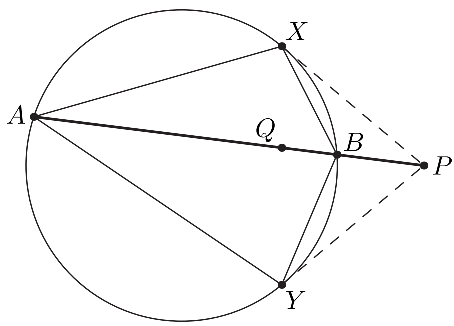

Figure 9.5B. Harmonic quadrilaterals again.

Proof. We consider only the case where P is outside ω and Q is inside it. Construct the tangents $\overline{PX}$ and $\overline{PY}$ to ω. Lemma 9.9 gives

$$(A, B; P, \overline{XY} \cap \overline{AB}) = -1,$$

so Q lies on the polar of P (namely line XY) if and only if $(A, B; P, Q) = -1$. □

We are now ready to state one of the most profound theorems about cyclic quadrilaterals. It shows that any cyclic quadrilateral has hidden within it three pairs of poles and polars.

Theorem 9.25 (Brocard's Theorem). *Let $ABCD$ be an arbitrary cyclic quadrilateral inscribed in a circle with center O, and set $P = \overline{AB} \cap \overline{CD}$, $Q = \overline{BC} \cap \overline{DA}$, and $R = \overline{AC} \cap \overline{BD}$. Then P, Q, R are the poles of QR, RP, PQ, respectively.*

In particular, O is the orthocenter of triangle PQR.

We say that triangle PQR is **self-polar** with respect to ω, because each of its sides is the polar of the opposite vertex.

† Not the best choice of terms, as the two are easily confused. Mnemonic: "pole" is shorter than "polar", and points are much smaller than lines.

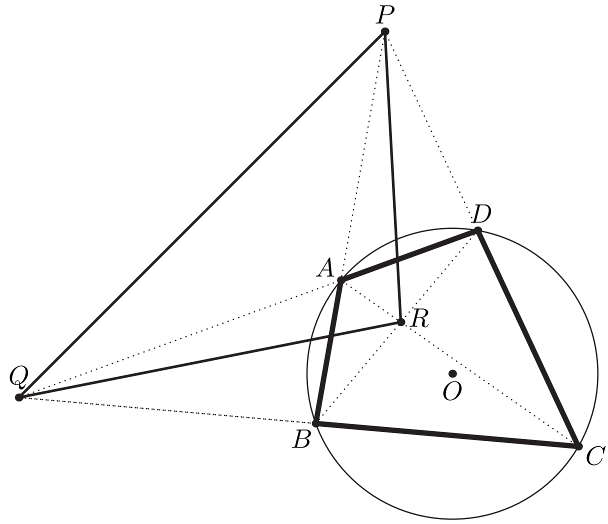

Figure 9.5C. The triangle PQR determined by completing a cyclic quadrilateral is self-polar.

Take a moment to appreciate the power of Brocard's theorem. Nowhere do the words "pole", "polar", "harmonic", "projective", or anything of that sort appear in the hypothesis. We could have stated this theorem in Chapter 1—all we did was take a completely arbitrary cyclic quadrilateral and intersect the sides and diagonals—and then suddenly, we have an entire orthocenter! It seems too good to be true. This really highlights the type of problems that projective geometry handles well: anything with lots of intersections and maybe a few circles.

On to the proof of the theorem. The idea is that Brocard's theorem looks a lot like Lemma 9.11.

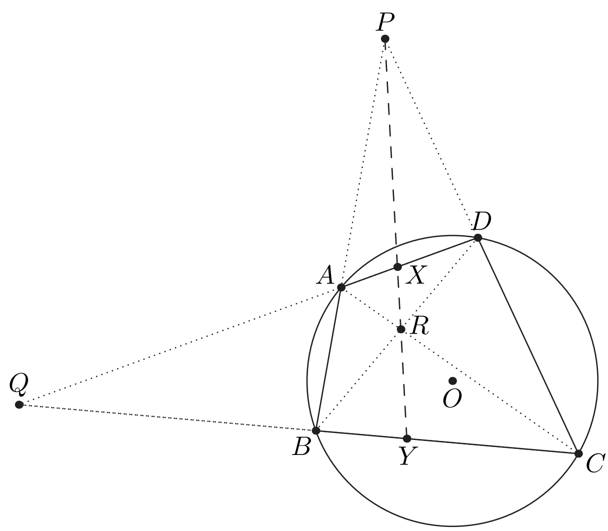

Figure 9.5D. Triangle PQR is self-polar.

Proof. First, we show that Q is the pole of line PR. Define the points $X = \overline{AD} \cap \overline{PR}$ and $Y = \overline{BC} \cap \overline{PR}$, as in Figure 9.5D. By Lemma 9.11, both $(A, D; Q, X)$ and $(B, C; Q, Y)$ are harmonic bundles.

Therefore, X and Y both lie on the polar of Q, by Proposition 9.24. Since the polar of Q is a line, it must be precisely line XY, which is the same as line PR.

The same can be used to show that P is the pole of line QR and R is the pole of line PQ; projective geometry is immune to configuration issues. (This is part of the reason we like points at infinity.) This gives that PQR is indeed self-polar. Finally, the definition of a polar implies that O is the orthocenter of triangle PQR, completing the proof. $\square$

Problems for this Section

Problem 9.26. Prove La Hire's theorem (Theorem 9.23).

Lemma 9.27 (Self-Polar Orthogonality). *Let ω be a circle and suppose P and Q are points such that P lies on the pole of Q (and hence Q lies on the pole of P). Prove that the circle γ with diameter $\overline{PQ}$ is orthogonal to ω.* **Hint**: 616

Problem 9.28. Let ABC be an acute scalene triangle, and let H be a point inside it such that $\overline{AH} \perp \overline{BC}$. Rays BH and CH meet $\overline{AC}$ and $\overline{AB}$ at E, F. Prove that if quadrilateral $BFEC$ is cyclic then H is in fact the orthocenter of ABC. **Hints**: 492 52

9.6 Pascal's Theorem

Pascal's theorem is of a different flavor than the previous theorems, but is useful in similar situations. It handles many points on a circle and their intersections. Here is the statement[‡]; see Example 7.27 for a proof. Many other proofs exist, of course.

Theorem 9.29 (Pascal's Theorem). *Let $ABCDEF$ be a cyclic hexagon, possibly self-intersecting. Then the points $\overline{AB} \cap \overline{DE}$, $\overline{BC} \cap \overline{EF}$, and $\overline{CD} \cap \overline{FA}$ are collinear.*

Note that Pascal's theorem can look very different depending on what order the vertices lie in. Figure 9.6A shows four different shapes that Pascal's theorem can take on. It is often useful to take two consecutive vertices of the hexagon to be the same point. The "side" AA degenerates to a tangent to the circle at A.[§] An example of this technique is in the solution to Example 9.38.

For an example, we revisit the first part of Lemma 4.40, and give a short proof using Pascal's theorem.

Example 9.30. Let ABC be a triangle inscribed in a circle. The A-mixtilinear circle is drawn, tangent to $\overline{AB}$, $\overline{AC}$ at K, L. Then the incenter I is the midpoint of $\overline{KL}$.

[‡] The converse is also true if we replace "circle" with "conic". See the next section on projective transformations.

[§] Think of it this way: $\overline{XY}$ is the line intersecting the circle at points X and Y. So $\overline{AA}$ is a line intersecting the circle at A and A, i.e., the tangent to A.

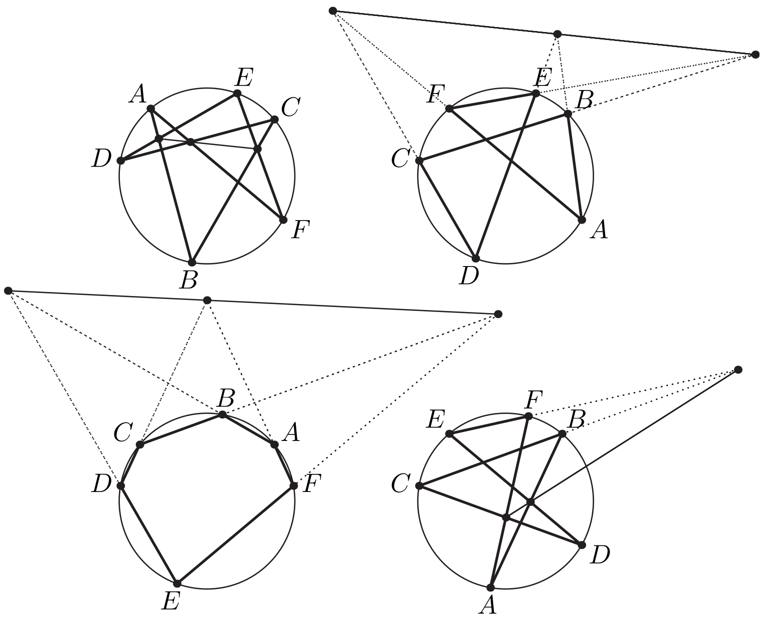

Figure 9.6A. The many faces of Pascal's theorem.

Proof. Obviously $\overline{AI}$ bisects $\overline{KL}$ (since $AK = AL$ and $\angle KAI = \angle IAL$) so it suffices to prove that K, I, L are collinear.

By Lemma 4.33, M_C, K, T are collinear, where M_C is the midpoint of arc AB not containing C. In particular, C, I, M_C are collinear. Similarly, the midpoint M_B of arc AC lies on both lines BI and LT. Now we just apply Pascal's theorem on the hexagon ABM_BTM_CC. □

An even more striking illustration is Problem 9.32 below.

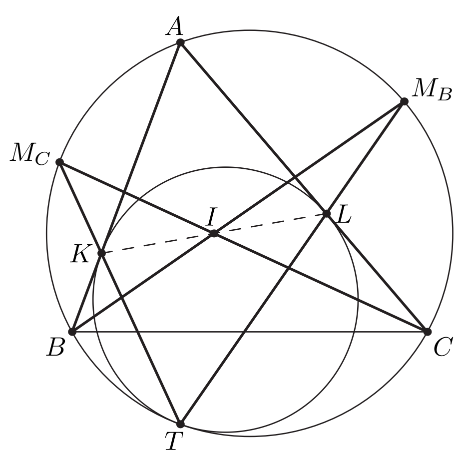

Figure 9.6B. Using Pascal's theorem on the A-mixtilinear incircle.

Problems for this Section

Problem 9.31. Let ABC be a triangle with circumcircle Γ. Let X be the intersection of line BC with the tangent to Γ at A. Define Y and Z similarly. Show that X, Y, Z are collinear. **Hint**: 378

Problem 9.32. Let $ABCD$ be a cyclic quadrilateral and apply Pascal's theorem to $AABCCD$ and $ABBCDD$. What do we discover? **Hints**: 421 473 309

9.7 Projective Transformations

This is only a brief digression on what is otherwise a deep topic. See the last chapter of [7] for further exposition.

Occasionally we run into a problem that we say is *purely projective*. Essentially this means the problem statement involves only intersections, tangency, and perhaps a few circles. This happens very rarely, but when it does, the problems can usually be eradicated via **projective transformations**.

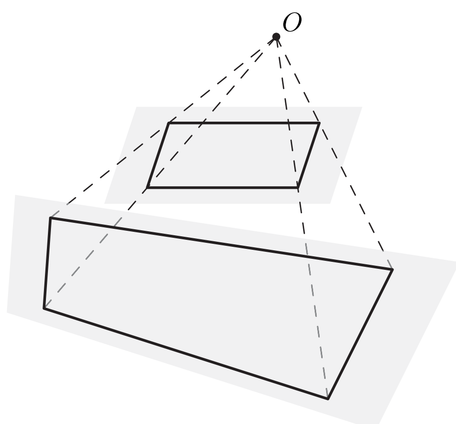

Figure 9.7A. An example of a projective transformation.

Projective transformations are essentially the most general type of transformation. Actually, they are defined as any map that sends lines to lines and conics to conics (but need not preserve anything else). Loosely speaking, a **conic** is a second-degree curve in the plane determined by five points. In more precise terms, a conic is a curve in the xy-plane of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

extended to include points at infinity. This includes parabolas, hyperbolas, and ellipses (in particular, circles). For our purposes, we only care that a circle is a conic. See Figure 9.7A.

Why would we consider a transformation that preserves so few things? The gain is encapsulated in the following theorem, stated without proof, which exploits the generality of the transformation.

Theorem 9.33 (Projective Transformations). *Each of the following is achievable with a unique projective transformation.*

(a) Taking four points A, B, C, D (no three collinear) to any other four points W, X, Y, Z (no three collinear).
(b) Taking a circle to itself and a point P inside the circle to any other point Q inside the circle.
(c) Taking a circle to itself and any given line outside the circle into the line at infinity.

Furthermore, projective transformations preserve the cross ratio of any four collinear points. Moreover, if four concyclic points are sent to four concyclic points, then the cross ratio of the quadrilaterals are the same.

The power of this technique is made most clear by example.

Example 9.34. Let $ABCD$ be a quadrilateral. Define the points $P = \overline{AD} \cap \overline{BC}$, $Q = \overline{AB} \cap \overline{CD}$, and $R = \overline{AC} \cap \overline{BD}$. Let X_1, X_2, Y_1, Y_2 denote $\overline{PR} \cap \overline{AD}$, $\overline{PR} \cap \overline{BC}$, $\overline{QR} \cap \overline{AB}$, $\overline{QR} \cap \overline{CD}$.

Prove that lines X_1Y_1, X_2Y_2, and PQ are concurrent.

This problem looks like a nightmare until we realize that it is purely projective. That means we can make some very convenient assumptions—we simply use a projective map taking $ABCD$ to a square $A'B'C'D'$.

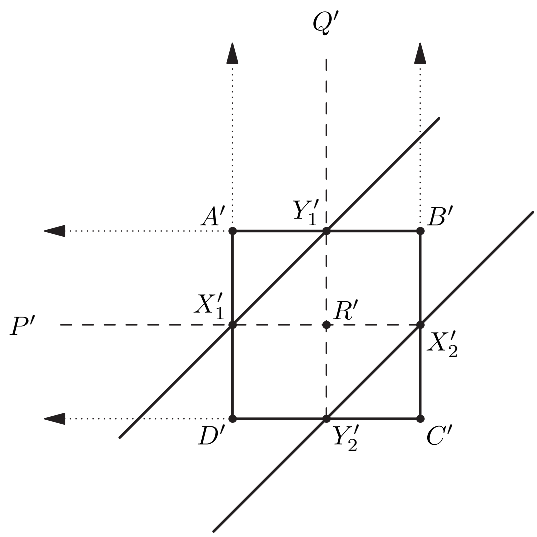

Figure 9.7B. We can take $ABCD$ to a square, trivializing the problem.

Solution. By Theorem 9.33, we can use a projective transformation to send $ABCD$ to the vertices of a square $A'B'C'D'$. Then P' is the intersection of lines $A'D'$ and $B'C'$, since projective transformations preserve intersections. We can define the remaining points similarly.

The problem is now trivial: just look at Figure 9.7B! P' and Q' become the points at infinity, and we find that X_1', X_2', Y_1', Y_2' are just midpoints of the respective sides. Hence

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the intersection of lines $\overline{X_1'Y_1'}$ and $\overline{X_2'Y_2'}$ is yet another point at infinity (as they are parallel). This implies P', Q', and $\overline{X_1'Y_1'} \cap \overline{X_2'Y_2'}$ are collinear along the line at infinity. $\square$

We can even extend this technique to tackle problems that do not look purely projective when the condition can be re-written with cross ratios. For example, consider the famous butterfly theorem.

Theorem 9.35 (Butterfly Theorem). *Let $\overline{AB}$, $\overline{CD}$, $\overline{PQ}$ be chords of a circle concurrent at M. Put $X = \overline{PQ} \cap \overline{AD}$ and $Y = \overline{PQ} \cap \overline{BC}$. If $MP = MQ$ then $MX = MY$.*

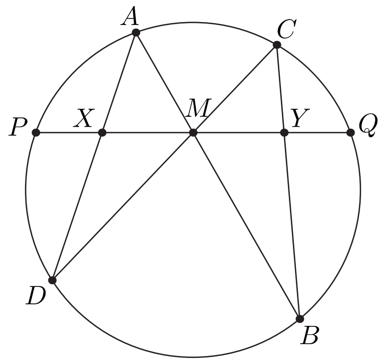

Figure 9.7C. The butterfly theorem.

Proof. This problem looks completely projective except for the midpoint condition. We can handle this by adding the point at infinity P_∞ to line PQ. The condition becomes $(P, Q; P_\infty, M) = -1$, and we wish to show that $(X, Y; P_\infty, M) = -1$.

By rewriting the givens as cross ratios, the problem becomes purely projective! We therefore take the projective transformation sending M to the center of the circle, say M'. Then $\overline{P'Q'}$ is a diameter. Because we must have the cross ratio $(P', Q', P'_\infty, M') = -1$ is preserved, we find that P'_∞ is still the point at infinity. Hence it simply suffices to prove that M' is the midpoint of $\overline{X'Y'}$.

On the other hand, proving the butterfly theorem when M is the center of the circle is not very hard. Actually, it is obvious by symmetry. Therefore $(X', Y', P'_\infty, M') = -1$. Consequently $(X, Y; P_\infty, M) = -1$ as well and we are done. $\square$

Problems for this Section

Problem 9.36. Give a short proof of Lemma 9.9 using projective transformations. **Hints**: 183 218 231

Problem 9.37. Give a short proof of Lemma 9.11 using projective transformations. **Hints**: 333 595

9.8 Examples

We present two example problems. First, let us consider the following problem from the 51st IMO.

Example 9.38 (IMO 2010/2). Let I be the incenter of a triangle ABC and let Γ be its circumcircle. Let line AI intersect Γ again at D. Let E be a point on arc $\widehat{BDC}$ and F a point on side BC such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. Finally, let G be the midpoint of $\overline{IF}$. Prove that $\overline{DG}$ and $\overline{EI}$ intersect on Γ.

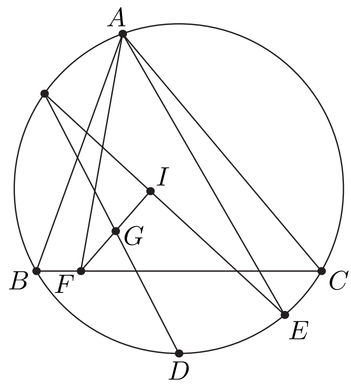

Figure 9.8A. Example 9.38.

We begin by extending $\overline{AF}$ to meet Γ again at a point F_1; evidently $\overline{F_1E} \parallel \overline{BC}$. We also let K denote the second intersection of $\overline{EI}$ with Γ. Our goal is to prove that $\overline{DK}$ bisects $\overline{IF}$.

Seeing so many points and intersections on a circle motivates us to try Pascal's theorem in the hopes of finding something interesting. Specifically, we have $I = \overline{AD} \cap \overline{KE}$, $\overline{DD} \cap \overline{EF_1}$ is the point at infinity, and $F = \overline{AF_1} \cap \overline{BC}$. Trying to string two of these into one application of Pascal's theorem, we find with some trial and error that the hexagon AF_1EKDD is useful.

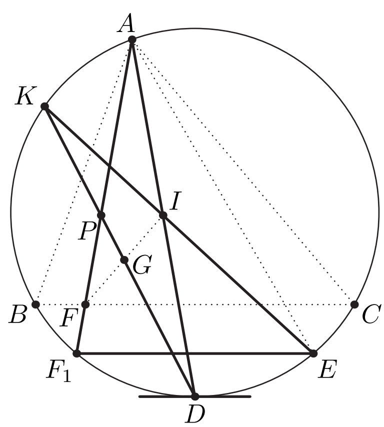

Figure 9.8B. Applying Pascal's theorem on Example 9.38.

Pascal's theorem now implies that $\overline{AF_1} \cap \overline{KD}$, the point at infinity $\overline{F_1E} \cap \overline{DD}$, and the incenter $I = \overline{DA} \cap \overline{KE}$ are collinear. In other words, if we set $P = \overline{AF_1} \cap \overline{KD}$, then we find that $\overline{IP} \parallel \overline{EF_1} \parallel \overline{BC}$.

Once the point P is introduced, we can effectively ignore the points E, F_1, and K now. In other words, we have the convenient recasting of the problem as follows.

> Let $\overline{AF}$ be a cevian of the triangle ABC and let P be a point on $\overline{AF}$ with $\overline{IP} \parallel \overline{BC}$. If D is the midpoint of arc $\widehat{BC}$ not containing A, then $\overline{DP}$ bisects $\overline{IF}$.

This is much simpler, and you can actually finish using barycentric coordinates. At least this indicates that we are probably on the right track. So what do we do next?

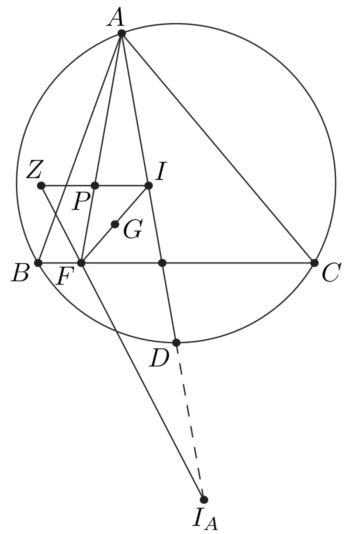

Figure 9.8C. The finishing touch using harmonic bundles.

Seeing the midpoint, we consider a homothety at I with ratio 2, which conveniently grabs the excenter I_A. That means it suffices to prove that if $Z = \overline{I_AF} \cap \overline{IP}$, then P is the midpoint of $\overline{IZ}$. Seeing midpoints and parallel lines once again, we take harmonic bundles (in light of Lemma 9.8). And indeed, the first decent choice of a point on $\overline{BC}$ works; perspectivity at F solves the problem.

Solution to Example 9.38. Let $\overline{EI}$ meet Γ again at K and $\overline{AF}$ meet Γ again at F_1. Set $P = \overline{DK} \cap \overline{AF}$ and $Z = \overline{IP} \cap \overline{I_AF}$. By Pascal's theorem on AF_1EKDD, we see that $\overline{IP} \parallel \overline{BC}$.

Setting I_A as the A-excenter and recalling Lemma 9.22 gives

$$-1 = \left(I, I_A; A, \overline{AI} \cap \overline{BC}\right) \stackrel{F}{=} \left(I, Z; P, \overline{BC} \cap \overline{IP}\right).$$

Since $\overline{IP} \parallel \overline{BC}$, we conclude that P is the midpoint of $\overline{IZ}$. Then we simply take a homothety at I. $\square$

Our other example is the final problem from an Asian-Pacific olympiad; it yields many different projective solutions. We present three of them.

Example 9.39 (APMO 2013/5). Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of $\overline{AC}$ such that $\overline{PB}$ and $\overline{PD}$ are tangent to ω. The tangent at C intersects $\overline{PD}$ at Q and the line AD at R. Let E be the second point of intersection between $\overline{AQ}$ and ω. Prove that B, E, R are collinear.

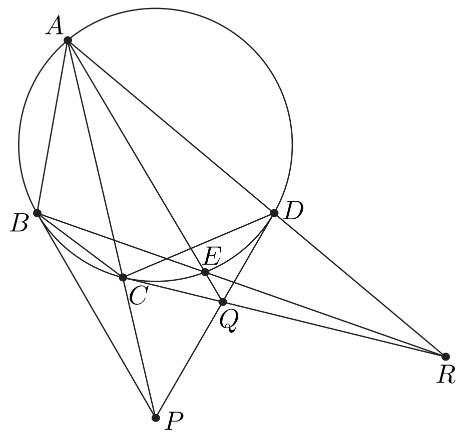

Figure 9.8D. Problem 5 from APMO 2013.

We immediately recognize Lemma 9.9 twice: $ACED$ and $ABCD$ are both harmonic quadrilaterals. This motivates us to try projective geometry in the first place, since there are a lot of intersections and the conditions are natural in the language of harmonic bundles.

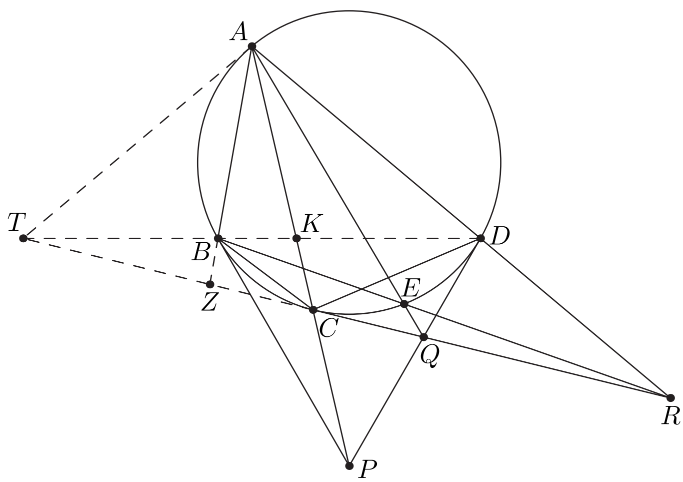

Figure 9.8E. A solution to Example 9.39 that involves only harmonic bundles.

In order to place things more in the frame of our projective tools, we let E' be the second intersection of line BR and ω. Then it would just suffice to prove $ACE'D$ is harmonic (rather than prove three points are collinear). How might we do that? We wish to prove that $(A, E'; C, D) = -1$. Are there any points that look good for projecting through on ω? After some trial we find that B looks like a good choice, because it handles the other points somewhat nicely, but more importantly it lets us deal with the point E'.

Because we again want to focus on making point E' behave well, we choose to project onto line CR.

So we find that

$$(A, E'; C, D) \stackrel{B}{=} (\overline{AB} \cap \overline{CR}, R; C, \overline{BD} \cap \overline{CR}).$$

Taking advantage of the fact that $ABCD$ is harmonic, we put $T = \overline{BD} \cap \overline{CR}$ as the intersection of the tangents at A and C (hence on line BD). The point T seems nice because it is pretty closely tied to $ABCD$.

On the other hand we should probably clean up $\overline{AB} \cap \overline{CR}$ in the next projection. Since we already took perspectivity from B, we try taking perspectivity from A this time (otherwise we are back where we started). Now the most logical choice for the line to project onto is BD. Letting $Z = \overline{AB} \cap \overline{CR}$ for brevity, we find

$$(Z, R; C, T) \stackrel{A}{=} (B, D; \overline{AC} \cap \overline{BD}, T).$$

But this is harmonic by Lemma 9.9. Hence with just two projections we are done.

Solution 1. Set $T = \overline{BD} \cap \overline{CR}$, $K = \overline{AC} \cap \overline{BD}$, $Z = \overline{AB} \cap \overline{CR}$ and let E' be the second intersection of $\overline{BR}$ with ω. Since $ABCD$ is harmonic, we have T, K, B, D collinear and therefore

$$-1 = (T, K; B, D) \stackrel{A}{=} (T, C; Z, R) \stackrel{B}{=} (D, C; A, E').$$

But $DACE$ is harmonic, so $E = E'$. □

A second solution involves interpreting the problem from the context of symmedians (see Lemma 4.26). We can view $\overline{DB}$ and $\overline{AE}$ as the symmedians of triangle ACD. Suddenly we can ignore the points P and Q completely! On the other hand we should probably add in the symmedian point K of triangle ACD, which is the intersection of $\overline{AE}$ and $\overline{BD}$.

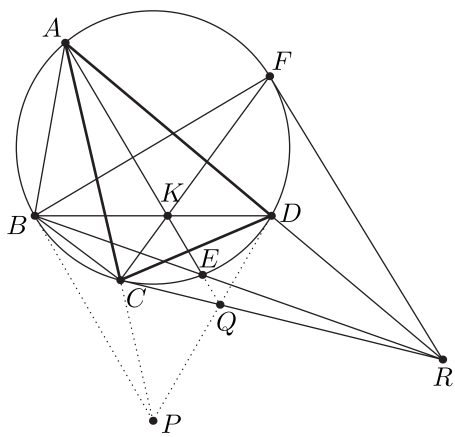

Figure 9.8F. Solving Example 9.39 using symmedians.

Now what of the point R? It is the intersection of the tangent at C with line AD. Trying to complete Lemma 9.9 again, we let F be the other point on ω other than C such that $\overline{RF}$

is a tangent. Hence $ACDF$ is harmonic. So $\overline{CF}$ is a symmedian as well. This completes the picture of the symmedian point. In particular, K lies on $\overline{CF}$.

Now for the finish. By Brocard's theorem, $\overline{BE} \cap \overline{AD}$ is the point on $\overline{AD}$ that lies on the polar of $K = \overline{BD} \cap \overline{AE}$. This is none other than the point R.

Solution 2. Let $K = \overline{AE} \cap \overline{BD}$ be the symmedian point of triangle ACD. Let F be the second intersection of ray CK with (ACD). Noticing the symmedians, we find three harmonic quadrilaterals $ACED$, $ABCD$, and $ACDF$.

In harmonic quadrilateral $ACDF$, we notice (by Lemma 9.9, say), that R is the pole of $\overline{CF}$. Because $\overline{CF}$ contains K, point R lies on the polar of K. Now by Brocard's theorem, the intersection of line BE with $\overline{AD}$ lies on the polar of K as well, implying that B, E, R are collinear. $\square$

Finally, one last solution—note this problem is purely projective!

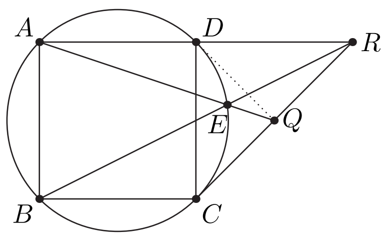

Figure 9.8G. Projective transformations trivialize Example 9.39, because they allow us to assume $ABCD$ is a square.

Take a projective transformation that fixes ω and sends the point $\overline{AC} \cap \overline{BD}$ to the center of the circle. Thus $ABCD$ is a rectangle. Because $ABCD$ is harmonic, it must in fact be a square. Thus P is the point at infinity along $\overline{AB} \parallel \overline{CD}$ and the problem is not very hard now.

9.9 Problems

Lemma 9.40 (Incircle Polars). *Let ABC be a triangle with contact triangle DEF and incenter I. Lines EF and BC meet at K. Prove that $\overline{IK} \perp \overline{AD}$.* **Hints**: 351 689 **Sol:** p.275

Theorem 9.41 (Desargues' Theorem). *Let ABC and XYZ be triangles in the projective plane. We say that the two triangles are **perspective from a point** if lines $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ concur (possibly at infinity), and we say they are **perspective from a line** if the points $\overline{AB} \cap \overline{XY}$, $\overline{BC} \cap \overline{YZ}$, $\overline{CA} \cap \overline{ZX}$ are collinear. Prove that these two conditions are equivalent.* **Hints**: 253 456

Problem 9.42 (USA TSTST 2012/4). In scalene triangle ABC, let the feet of the perpendiculars from A to $\overline{BC}$, B to $\overline{CA}$, C to $\overline{AB}$ be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the

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respective midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Show that the perpendiculars from D to $\overline{AA_2}$, E to $\overline{BB_2}$, and F to $\overline{CC_2}$ are concurrent. **Hints**: 308 233

Problem 9.43 (Singapore TST). Let ω and O be the circumcircle and circumcenter of right triangle ABC with $\angle B = 90^\circ$. Let P be any point on the tangent to ω at A other than A, and suppose ray PB intersects ω again at D. Point E lies on line CD such that $\overline{AE} \parallel \overline{BC}$. Prove that P, O, and E are collinear. **Hints**: 587 675

Problem 9.44 (Canada 1994/5). Let ABC be an acute triangle. Let $\overline{AD}$ be the altitude on $\overline{BC}$, and let H be any interior point on $\overline{AD}$. Lines BH and CH, when extended, intersect $\overline{AC}$, $\overline{AB}$ at E and F respectively. Prove that $\angle EDH = \angle FDH$. **Hints**: 20 164 80 **Sol:** p.275

Problem 9.45 (Bulgarian Olympiad 2001). Let ABC be a triangle and let k be a circle through C tangent to $\overline{AB}$ at B. Side $\overline{AC}$ and the C-median of $\triangle ABC$ intersect k again at D and E, respectively. Prove that if the intersecting point of the tangents to k through C and E lies on the line BD then $\angle ABC = 90^\circ$. **Hints**: 111 318 571

Problem 9.46 (ELMO Shortlist 2012). Let ABC be a triangle with incenter I. The foot of the perpendicular from I to $\overline{BC}$ is D, and the foot of the perpendicular from I to $\overline{AD}$ is P. Prove that $\angle BPD = \angle DPC$. **Hints**: 240 354 347 **Sol:** p.276

Problem 9.47 (IMO 2014/4). Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC. **Hints**: 145 216 286 **Sol:** p.276

Problem 9.48 (Shortlist 2004/G8). Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = \overline{AC} \cap \overline{BD}$ and $F = \overline{BC} \cap \overline{DA}$. **Hints**: 58 503 632

Problem 9.49 (Sharygin 2013). The incircle of triangle ABC touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points A', B', and C' respectively. The perpendicular from the incenter I to the C-median meets the line $A'B'$ in point K. Prove that $\overline{CK} \parallel \overline{AB}$. **Hint**: 55 **Sol:** p.277

Problem 9.50 (Shortlist 2004/G2). Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let $\overline{AB}$ be a diameter of the circle Γ; assume that this diameter $\overline{AB}$ is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same half-plane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A.

Prove that the reflection of the point G in the line AB lies on the line CF. **Hints**: 25 285 406 497 **Sol:** p.277

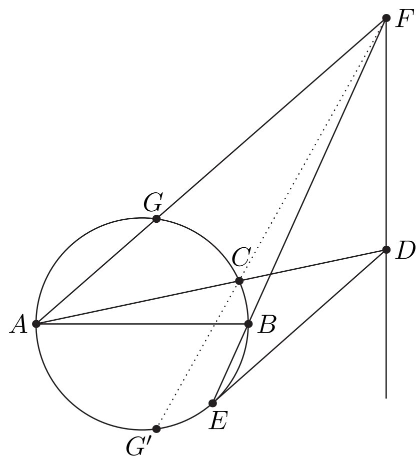

Figure 9.9A. Problem 9.50 is a mouthful.

Problem 9.51 (USA January TST for IMO 2013). Let ABC be an acute triangle. Circle ω_1, with diameter $\overline{AC}$, intersects side $\overline{BC}$ at F (other than C). Circle ω_2, with diameter $\overline{BC}$, intersects side $\overline{AC}$ at E (other than C). Ray AF intersects ω_2 at K and M with $AK < AM$. Ray BE intersects ω_1 at L and N with $BL < BN$. Prove that lines AB, ML, NK are concurrent. **Hints**: 168 374 239

Problem 9.52 (Brazilian Olympiad 2011/5). Let ABC be an acute triangle with orthocenter H and altitudes $\overline{BD}$, $\overline{CE}$. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on $\overline{BC}$. **Hints**: 405 221 366

Problem 9.53 (ELMO Shortlist 2013). In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to $\overline{BC}$. **Hints**: 511 34 270

Problem 9.54 (APMO 2008/3). Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides $\overline{BC}$ and $\overline{BA}$ at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines to Γ at A and C meet the line DE at L and M, respectively.

Prove that the lines LH and MG meet at Γ. **Hints**: 156 444 352 572 **Sol:** p.277

Theorem 9.55 (Brianchon's Theorem). *Let $ABCDEF$ be a hexagon circumscribed about a circle ω. Prove that $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are concurrent.* **Hints**: 241 35

Problem 9.56 (ELMO Shortlist 2014). Suppose $ABCD$ is a cyclic quadrilateral inscribed in the circle ω. Let $E = \overline{AB} \cap \overline{CD}$ and $F = \overline{AD} \cap \overline{BC}$. Let ω_1, ω_2 be the circumcircles of triangles AEF, CEF, respectively. Let G and H be the intersections of ω and ω_1, ω and ω_2, respectively, with $G \neq A$ and $H \neq C$. Show that $\overline{AC}$, $\overline{BD}$, and $\overline{GH}$ are concurrent. **Hints**: 404 590 443 **Sol:** p.278

Problem 9.57 (ELMO Shortlist 2014). Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω. The tangent to ω at A intersects lines CD and BC at E and F. Lines BE and DF meet ω again G and I, and $H = \overline{BE} \cap \overline{AD}$, $J = \overline{DF} \cap \overline{AB}$. Prove that $\overline{GI}$, $\overline{HJ}$, and the B-symmedian of $\triangle ABC$ are concurrent. **Hints**: 667 234

Problem 9.58 (Shortlist 2005/G6). Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to BC, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$. **Hints**: 682 543 328 104 563